P602

MATHEMATICAL SUBROUTINES

Order #E7675.1

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ELEMENTARY FUNCTIONS - ROM Routines

There are 14 special routines programmed in the ROM, shown in the following chart :

ROM Routine	Jump required	Argument	Result	Register used
$Log_{10}x$	Y	in A	in A	M, A, R, B, b
Ln x	AY			
10^x	Z		man in A, exp in M	
e^x	AZ			
ln 10	AV		in M	
$\pi/2$	AW			
sin x	RV	in A	in A	M, A, R, B,
cos x	RW			
tan x	RY			M, A, R, B, b
arctan x	RZ			
Floating +	BV	first operand in B; second operand in A	in A	M, A, R, B, b
Floating -	BW			
Floating X	BY			
Floating ÷	BZ			

In addition to there there are two other calls to the ROM that are useful:

FW - for evaluating cos $(\frac{\pi}{2} x)$ (see discussion in section 20. 1)

W - for normalizing an arbitrary floating point number (see discussion in section 20. 4)

Actually, any label appearing in the ROM may be called, but attention must be paid to the status of the ROM sense switches in determining their usefulness.

Because there are so many common parts to the particular set of routines programmed in the ROM, the 14 routines are grouped into four sections here for explanation, plus a section showing the program instructions for the complete ROM:

20.1 Cos x, Sin x, Tan x, $\pi/2$

20.2 Arctan x

20.3 10^x, e^x, log x, ln x, ln 10

20.4 FL+, FL-, FLX, FL÷

20.5 Complete ROM

The program instructions for the main computations performed are given under their respective sections. In each section there is also a flow chart to show the logical flow among the various parts of the routines, Jumps, labels and sense switches used are indicated on the flow charts.

Notes:

- The sense switches used in the ROM are different from those in the main memory although of the same names. Whenever /* is executed in the main program, all the ROM sense switches are automatically reset (turned OFF).
- All constants used in the ROM routines are generated using the A ↑ instruction.
- Only registers B and b are used as working registers for the ROM routines.
- Return Jump is coded R✓ in the ROM routines.
- All trig functions deal with angles expressed in radians.

Title
Cos x, Sin x, Tan x, $\pi/2$

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

Cos x

This is the basic routine from which the sin x and tan x are also determined. The angle x must be given in radians and is first normalized by the following series of transformations:

1.) $\xi = \frac{x}{\pi/2} = \frac{2x}{\pi}$ yields the number of quadrants contained in the angle (angle may represent any number of revolutions.)

where

ξ = QUADRANTS

X = RADIANS

cos - II	cos + I
cos - III	cos + IV

2.) r = fractional part of $\left\{\frac{\xi}{4}\right\}$. After integer division by 4, the remainder must lie between 0 and 4. The integer part of this remainder then represents the quadrant minus 1. Since $\cos\theta = \cos(2n\pi + \theta)$ this reduces the angle to:

$$-2\pi < \theta < 2\pi$$

3.) $u = |r|$ Since $\cos\theta = \cos|\theta|$, this further reduces the angle to

$$0 \leq \theta < 2\pi$$

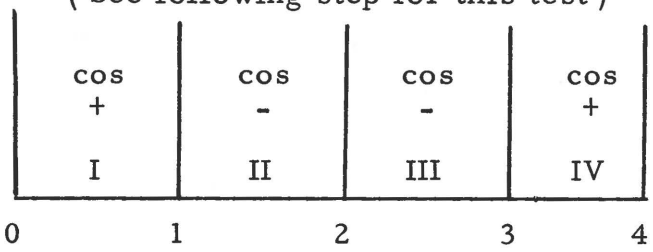
4.) $v = u - 2$ yields $-\cos\theta$ since $-\cos\theta = \cos(\theta - \pi)$.
(In the units of v now obtained, 2 is equivalent to π).

5.) $w = |v|$ Since $-\cos\theta = -\cos|\theta|$ the angle now satisfies $0 \leq \theta \leq \pi$.
This operation folds the plane such that

$|w| > 1$ ⇒ a positive result
(quadrants I and IV)

$|w| \leq 1$ ⇒ a negative result
(quadrants II and III)

(See following step for this test)



6.) $y = |w - 1| - 1 = \begin{cases} -w \text{ if } w < 1 \\ w - 2 \text{ if } > 1 \end{cases}$

A sense switch is set to change the sign of the final answer if $w \leq 1$. This transformation is allowable since

$$\cos = \cos(-\theta)$$
$$\text{and } \cos = -\cos(\theta - \pi)$$

The angle now satisfies:

$$-\frac{\pi}{2} \leq \theta \leq 0$$

so that $|y|$ $(0 \leq y \leq 1)$ represents the original angle or a fraction of one quadrant.

7.) A Chebychev series for $\cos(\frac{\pi}{2}y)$ where $(-1 \leq y \leq 1)$ is now evaluated and the sign adjusted for quadrant. For scaling purposes the square of the argument is divided by 10 and the coefficients adjusted accordingly:

$$\cos(\tfrac{\pi}{2}y) = a_0 + \frac{y^2}{10}(a_2 + \frac{y^2}{10}(a_4 + \frac{y^2}{10}(a_6 + \frac{y^2}{10}(a_8 + \frac{y^2}{10}a_{10}))))$$

$$\begin{aligned} a_0 &= 1 \\ a_2 &= -12.337\ 005\ 343 \\ a_4 &= 25.366\ 932\ 25 \\ a_6 &= -20.862\ 687\ 5 \\ a_8 &= 9.177\ 239 \\ a_{10} &= -2.382\ 45 \end{aligned}$$

$0 \leq y^2 \leq 1$

original coefficients are:

$a'_{2i} = a_{2i} \cdot 10^{-i}$

In summary, we then have

$$\cos x = \cos(\tfrac{\pi}{2}y) \doteq \sum_{i=0}^{10} \alpha_i y^{2i}$$

except for sign which is adjusted as shown above. Here x is the original argument in radians and y is the final transformed argument used in the series.

The theoretical error using the above series is of the order $2 \cdot 10^{-10}$.

The routine is:

$\left.\begin{matrix}\pi/2 \text{ in M} \\ x \text{ in A}\end{matrix}\right\}$

```
275  ÷
     A↑
     d+
     /÷
     R↓
     A↓
     A↑
     d↓
     -
     -
     A↓
     -
```

Transformations in text:

$$s = \left| \left| \left\{ \frac{\left(\frac{2x}{\pi}\right)}{4} \right\}_{fract} \right| -2 \right| -1$$

$> 0 \Rightarrow$ positive result

$\leq 0 \Rightarrow$ negative result

(In the text: $s = |w-1|$)

if $s > 0$: set final result positive
$s \leq 0$: set final result negative

```
     A↓
     -
     AX
     A↑
     f↓
     aX
297  B↕
```

Then $y = |s| - 1$

$\frac{y^2}{10}$ is now stored in B

[The Series is now evaluated]

Cos ($\frac{\pi}{2}$ x)

Advantage may be taken of part of the normalization for the cosine routine so that, given an argument x that is in units of $\pi/2$, the cosine of the angle represented may be obtained. The sequence uses a jump to the ROM Label FW:

(A) = x $\qquad (-1 \leq x \leq 1)$

```
*
/*      } Jump to ROM FW
FW
A↕      change sign
```

clear M

cos ($\frac{\pi}{2}$ x) is left in the Accummulator.

Sin x

The relation

$$\text{Sin } x = \text{Cos } (\frac{\pi}{2} - x)$$

is used for this function.

Tan x

The tangent is obtained simply as the ratio of the sine and cosine using the previous routines:

$$\text{Tan } (x) = \frac{\text{Sin } (x)}{\text{Cos } (x)}$$

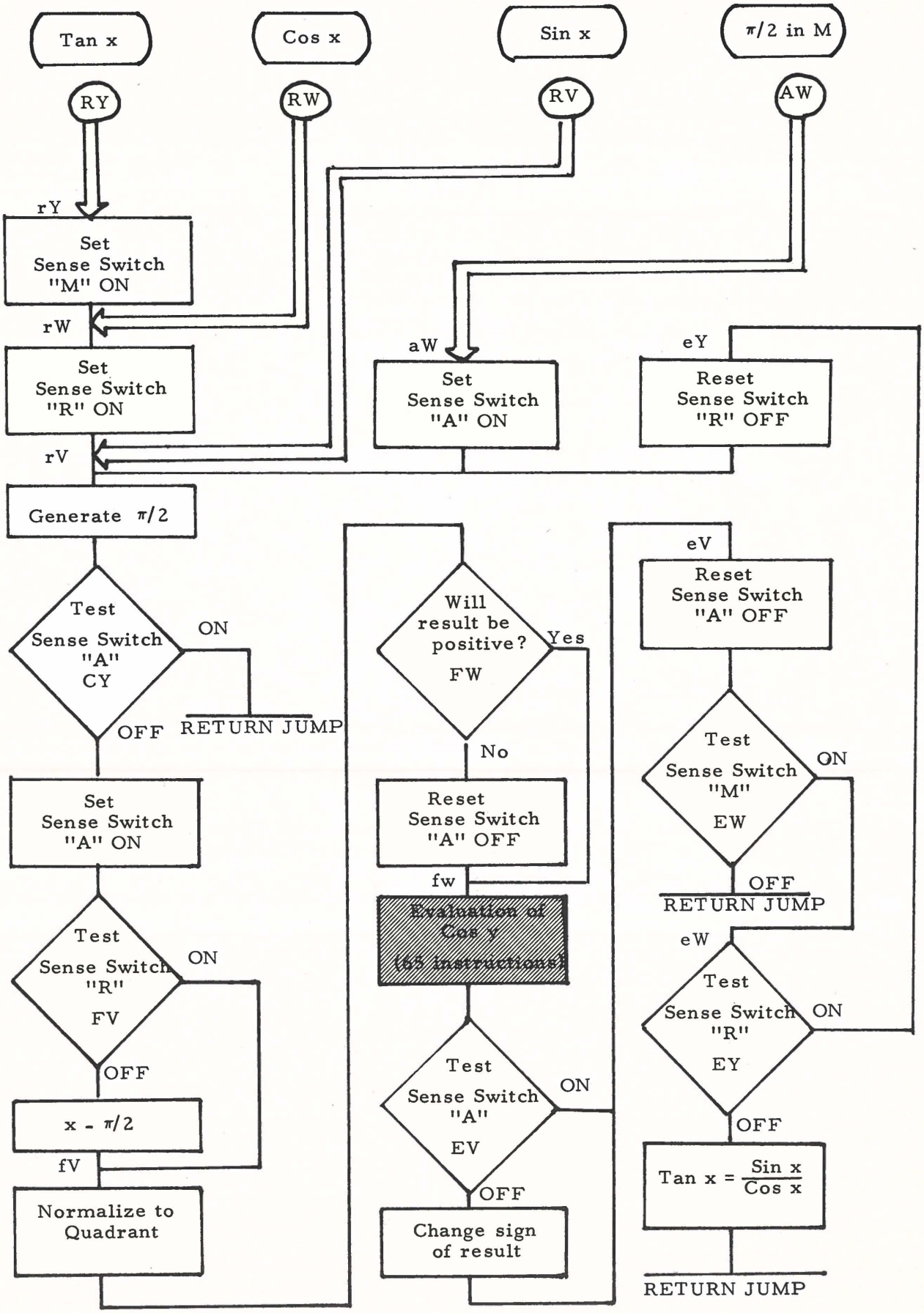
The error for values of the tangent less than 1 is of the order $\pm 4 \cdot 10^{-10}$.

For values of tan (x) greater than 1 the error increases although the relative error remains under $5 \cdot 10^{-10}$.

Series for evaluating cos ($\frac{\pi}{2}$ y)

1	A↑
2	f↓
3	aX
4	B↕
5	A↑
6	B−
7	B+
8	B↑
9	B◊
10	B↕
11	f↑
12	↓
13	BX
14	A↑
15	R*
16	R↕
17	R↑
18	R÷
19	R÷
20	R↓
21	d*
22	+
23	BX
24	A↑
25	B−
26	B÷
27	B◊
28	BX
29	B↑
30	BX
31	B◊
32	FS
33	b↑
34	+
35	BX
36	A↑
37	R−
38	R↑
39	R↑
40	R↕
41	R*
42	RX
43	RX
44	R↕
45	D−
46	r↑
47	+
48	BX
49	A↑
50	B↕
51	B+
52	B↕
53	B−
54	BS
55	BS
56	B÷
57	B↕
58	B↕
59	F↑
60	b↓
61	+
62	BX
63	A↑
64	d↓
65	+

This sequence corresponds to the shaded box in the flow chart. The first instruction here corresponds to instruction 294 in the complete ROM chart.


Tan x
Cos x
Sin x
π/2 in M
RY
RW
RV
AW
rY
Set Sense Switch "M" ON
rW
Set Sense Switch "R" ON
rV
aW
Set Sense Switch "A" ON
eY
Reset Sense Switch "R" OFF
Generate π/2
Test Sense Switch "A" CY
ON
RETURN JUMP
OFF
Set Sense Switch "A" ON
Test Sense Switch "R" FV
ON
OFF
x - π/2
fV
Normalize to Quadrant
Will result be positive? FW
Yes
No
Reset Sense Switch "A" OFF
fw
Evaluation of Cos y (65 instructions)
Test Sense Switch "A" EV
ON
OFF
Change sign of result
eV
Reset Sense Switch "A" OFF
Test Sense Switch "M" EW
ON
OFF
RETURN JUMP
eW
Test Sense Switch "R" EY
ON
OFF
Tan x = Sin x / Cos x
RETURN JUMP

Title
Arctan x

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

The evaluation of the arctangent (principal value) of an argument is based on a Chebychev series: If the argument is positive, the result will be negative; if it is negative, the result will be negative:

$$\arctan x = \frac{\pi}{8} + z\left(a_1 + \frac{z^2}{10}\left(a_3 + \frac{z^2}{10}\left(a_7 + \frac{z^2}{10}\left(a_9 + \frac{z^2}{10}\, a_{11}\right)\right)\right)\right)$$

where
$\pi/8 = 0.392\ 699\ 0817$
$a_1 = 0.414\ 213\ 5608$
$a_3 = -0.236\ 892\ 255$
$a_5 = 0.243\ 829\ 38$
$a_7 = -0.297\ 563\ 9$
$a_9 = 0.376\ 074$
$a_{11} = -0.359\ 1$

and the transformations are:

$$z = \frac{(\sqrt{2}+1)\,y}{(\sqrt{2}-1)\,y} \frac{-1}{+1}$$

$$y = \begin{cases} |x| & \text{if } |x|<1 \\ \dfrac{1}{|x|} & \text{if } |x|>1 \end{cases}$$

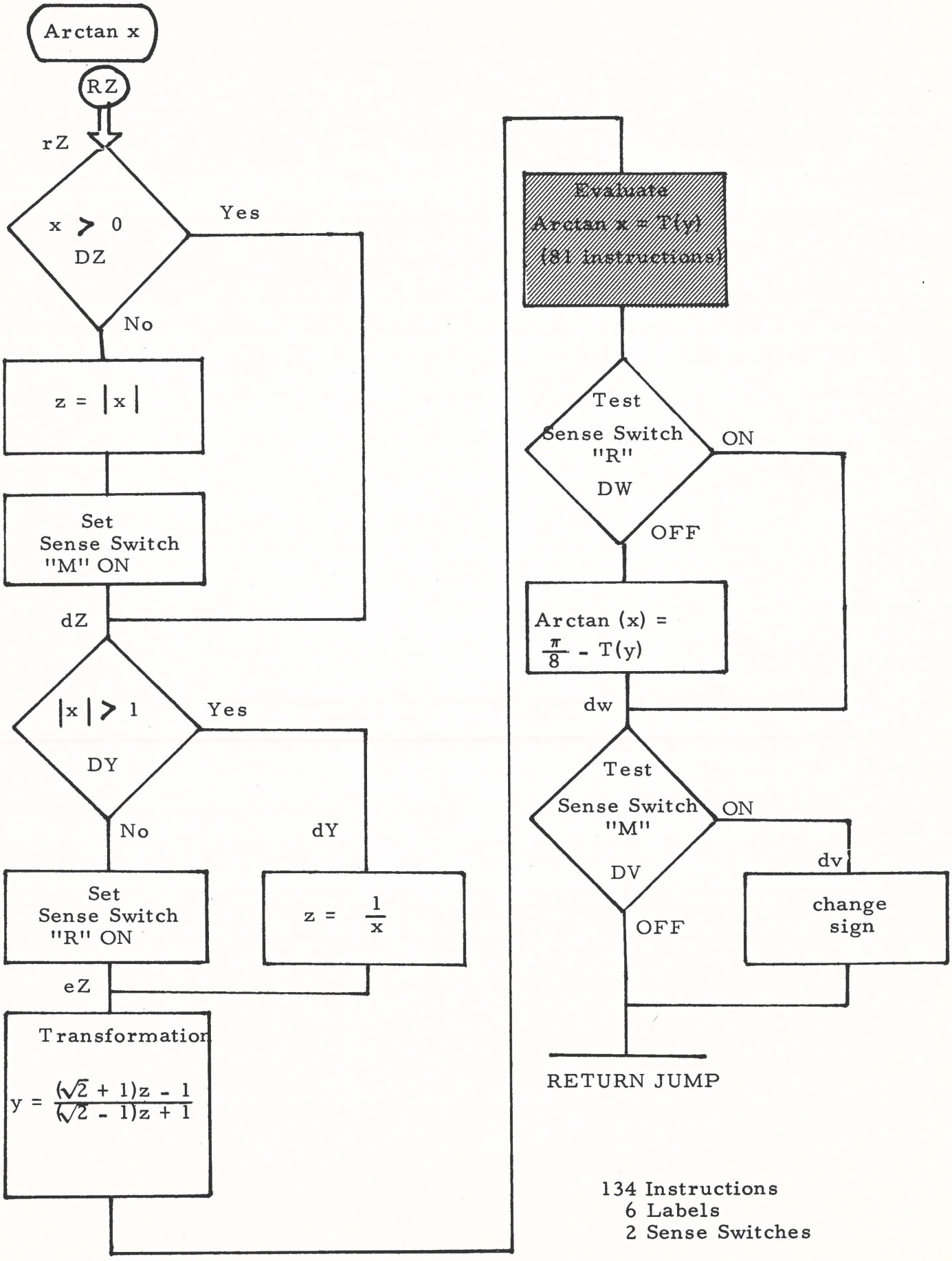
Series for evaluating Arctan (x)

1	A↑
2	B↓
3	B*
4	B−
5	B↕
6	fS
7	↓
8	BX
9	A↑
10	R+
11	R÷
12	RS
13	RX
14	R÷
15	R↕
16	dS
17	+
18	BX
19	A↑
20	B*
21	B↕
22	BX
23	B−
24	B÷
25	B*
26	B↑
27	fS
28	+
29	BX
30	A↑
31	R◊
32	R↕

33	R*
34	R↑
35	R◊
36	R↕
37	R+
38	R↑
39	dS
40	+
41	BX
42	A↑
43	B−
44	B−
45	B↑
46	B↑
47	B*
48	B◊
49	BX
50	B↕
51	B↑
52	fS
53	+
54	BX
55	A↑
56	R◊
57	RS
58	RX
59	R−
60	R↕
61	R↓
62	R↑
63	R+
64	R↓

65	R+
66	dS
67	+
68	bX
69	A↑
70	R÷
71	R↓
72	R◊
73	RS
74	R*
75	R*
76	RX
77	R↑
78	R*
79	R↕
80	dS
81	+

This sequence corresponds to the shaded box in the flow chart. The first instruction here corresponds to instruction 413 in the complete ROM chart.



134 Instructions
6 Labels
2 Sense Switches

Title
10^x, e^x, Log x, Ln x, Ln 10

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

Normalization of Argument of Log_{10} X and Ln X

In evaluating algorithms, the given argument is first normalized such that the significant figures are represented as a decimal fraction:

$$X = x \cdot 10^E \qquad , \ 0.1 \leq x < 1$$

Log_{10} X

The evaluation of the common logarithm of the absolute value of the normalized argument x is based on a Chebychev series:

$$Log_{10} X = w\,(a_1 + w^2\,(a_3 + w^2\,(a_5 + w^2\,(a_y))))$$

$$\text{where } w = \frac{\sqrt[8]{x} - 1}{\sqrt[8]{x} + 1}$$

$a_1 = 6.948\ 711\ 705\ 5$

$a_3 = 2.316\ 240\ 44$

$a_5 = 7.389\ 18$

$a_7 = 1.029\ 7$

The theoretical error in using this series is of the order 10^{-10}.

After this evaluation, the exponent from the normalization is added:

$$Log_{10}|X| = Log_{10}|x| + E$$

Ln X

The natural logarithm of the absolute value of an argument is taken from the evaluation of the common logarithm:

$$\text{Ln}|X| = \text{Ln } 10 \; \text{Log}_{10}|X|$$

where Ln 10 = 2.302 585 093

e^x (Natural Antilog)

The evaluation of the exponential to the base e ($\doteq$ 2.7182 81828) is determined from the exponential to the base 10 by the following relationship:

$$e^x = 10^{(x \; \text{Log}_{10} e)} = 10^{\left(\frac{x}{\text{Ln } 10}\right)}$$

Normalization of Argument for 10^x

The given argument is first normalized by separating the integer and fractional parts:

$$X = E + x, \qquad 0 \leq x < 1$$

10^x (Common Antilog)

The evaluation of the exponential to the base 10 of the normalized argument is based on a Chebychev Series:

$$10^x = \left(1 + \frac{x}{10}\left(a_1 + \frac{x}{10}\left(a_2 + \frac{x}{10}\left(a_3 + \frac{x}{10}\left(a_4 + \frac{x}{10}\left(a_5 + \frac{x}{10}\left(a_6\right)\right)\right)\right)\right)\right)\right)^{16}$$

where $a_1 = 1.439\ 115\ 683\ 4$

$a_2 = 1.035\ 526\ 975$

$a_3 = 0.496\ 747\ 48$

$a_4 = 0.178\ 719\ 3$

$a_5 = 0.051\ 484$

$a_6 = 0.012\ 35$

The theoretical error in using this series is of the order $10^{x/16} \leq 0.4 \cdot 10^{-11}$

The result of this evaluation is left in the accummulator while the integer part of the original argument is placed in the M register. In this way, either a fixed point or floating point result may be obtained using the following instructions, respectively:

fixed point	floating point
ax shift left or right	/↕ join exponent and mantissa

Note

In the case of X being a pure integer, the mantissa of the result for the routine 10^X will not be normalized, so this normalization must be performed before joining exponent and mantissa with the instruction /↕ , should a floating point result be desired. Both the normalization and joining may be accomplished by calling the ROM Label W (see (see NORMALIZE FLOATING POINT NUMBER in section 20.4). Since $e^X = 10^{X/Ln\ 10}$, the same procedure should be used for e^X, if X is an exact multiple of the constant Ln 10 = 2.302585093 used in the e^X routine.

Example

10^{11}

```
11    ↓
 /*
  Z
 /*
  W
12 100000  e◊
```

$e^{6Ln\ 10} = e^{13.815510558}$

```
13.815510558    ↓
                /*
                AZ
                /*
                 W
7 1000000000  e◊
```

Series for evaluating Log_{10} x

1	A √
2	A √
3	A √
4	+
5	B↕
6	B↓
7	−
8	−
9	B+
10	AX
11	B↑
12	A↑
13	R↓
14	R*
15	R↑
16	RS
17	d↓
18	↕
19	X
20	A↑
21	R◊
22	R↓
23	R*
24	R◊
25	R↕
26	d↓
27	+
28	BX
29	X
30	A↑
31	R+
32	R+

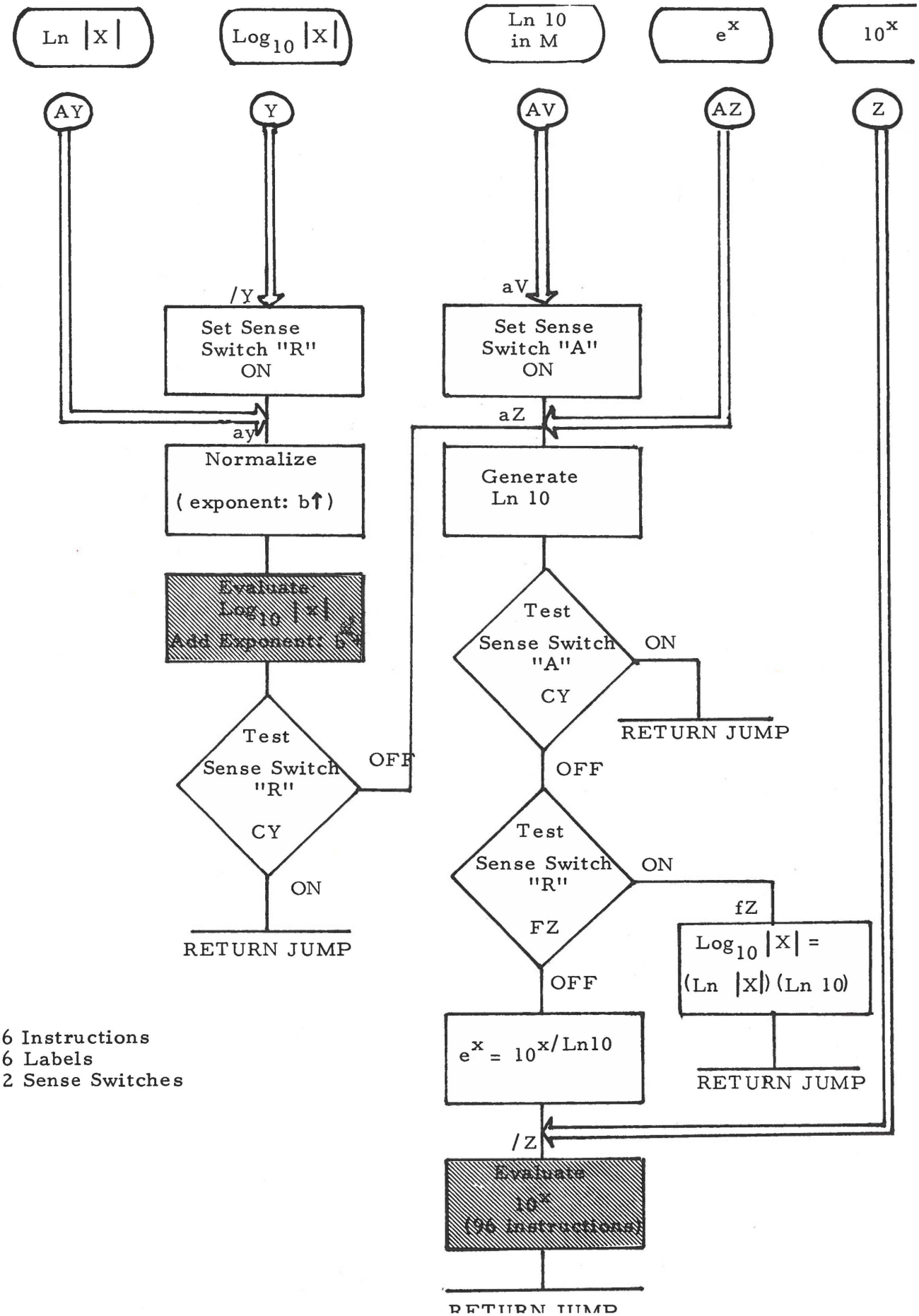
33	RS
34	R+
35	R↑
36	RX
37	R↓
38	R↕
39	d↑
40	+
41	BX
42	X
43	A↑
44	R−
45	R−
46	RS
47	R+
48	R↓
49	R↓
50	R+
51	R◊
52	R+
53	R*
54	dX
55	+
56	BX

This sequence corresponds to the shaded box for Log_{10} x shown in the flow chart. The first instruction corresponds to instruction 188 in the complete ROM chart.

Series for evaluating 10^x:

1	r↕	33	R−	65	RX
2	b↑	34	RS	66	R↑
3	A↑	35	dS	67	R−
4	f↓	36	+	68	R−
5	+	37	BX	69	R↕
6	b↕	38	A↑	70	RS
7	−	39	R↕	71	d↓
8	b↕	40	R*	72	+
9	r↕	41	R↓	73	BX
10	B↕	42	R+	74	A↑
11	b↓	43	R◊	75	R+
12	+	44	R÷	76	R↕
13	b↕	45	R↓	77	R◊
14	B↓	46	dS	78	RX
15	A↑	47	+	79	R−
16	f↓	48	BX	80	R↓
17	aX	49	A↑	81	R↓
18	B↕	50	R◊	82	R*
19	A↑	51	R+	83	R↕
20	R−	52	R÷	84	R+
21	R↕	53	R+	85	d↓
22	R↑	54	R+	86	+
23	R↓	55	RX	87	BX
24	RS	56	R*	88	A↑
25	dS	57	R+	89	d↓
26	↓	58	dS	90	+
27	BX	59	+	91	AX
28	A↑	60	BX	92	AX
29	R+	61	A↑	93	AX
30	R◊	62	R−	94	AX
31	R+	63	R+	95	b+
32	R↓	64	R*	96	−

This sequence corresponds to the shaded box for 10^x in the flow chart. The first instruction corresponds to instruction 83 in the complete ROM chart.



186 Instructions
6 Labels
2 Sense Switches

Title
Floating + , Floating - , Floating X , Floating ÷

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

The ROM provides the elementary operations of addition, subtraction, multiplication and division upon operands expressed in floating point. These operands must be placed in the B and A registers; at the end of the operation the original contents of B and A are destroyed and the result is left in A. For the non-communative operations of subtraction and division the sense of the operation is defined as:

$$(A) - (B) \rightarrow A$$

$$\frac{(A)}{(B)} \rightarrow A$$

The maximum number of places in the mantissa of the result is 14, and if the upper decimal wheel is set to FL, then 14 decimal places will result. However, when joined to an exponent of more than one digit, the result cannot be stored in a half register.

Operations on Signs

Both the signs of the exponent and the mantissa may be operated upon :

Scientific Notation: $0.mmmm \times 10^{c}$

Floating Print : c mmmm

Operations on Signs:

A↕ : change sign of contents of Accumulator, therefore of both c(exponent) and mmmm (mantissa)

a↕ : change sign of decimal part of contents of accumulator, therefore only mmmm (mantissa)

```
-5.12345    ↓
-5-12345    a◊   Print Floating
            A↕   Change sign of both exponent and mantissa
 5 12345    a◊   Print Floating
            a↕   Change sign of Mantissa
 5-12345    a◊   Print Floating
```

NORMALIZE FLOATING POINT NUMBER

An arbitrary floating point number may be normalized by calling the ROM Label W: (Exponent must be in register M and b; mantissa in A)

Normalize 20.456×10^{5}

```
20.456    ↓
      5   b↑
          /*
          W
7 20456   a◊
```

Normalize 123.987×10^{-7}

```
123.987   ↓
     -7   b↑
          /*
          W
-4 12398  a◊
```

The sequence performed by this call is:

```
a÷
b↕
+
↕
b↓
/↕
```

Note that the instruction a÷ normalizes a fixed point number in preparation for floating point representations.

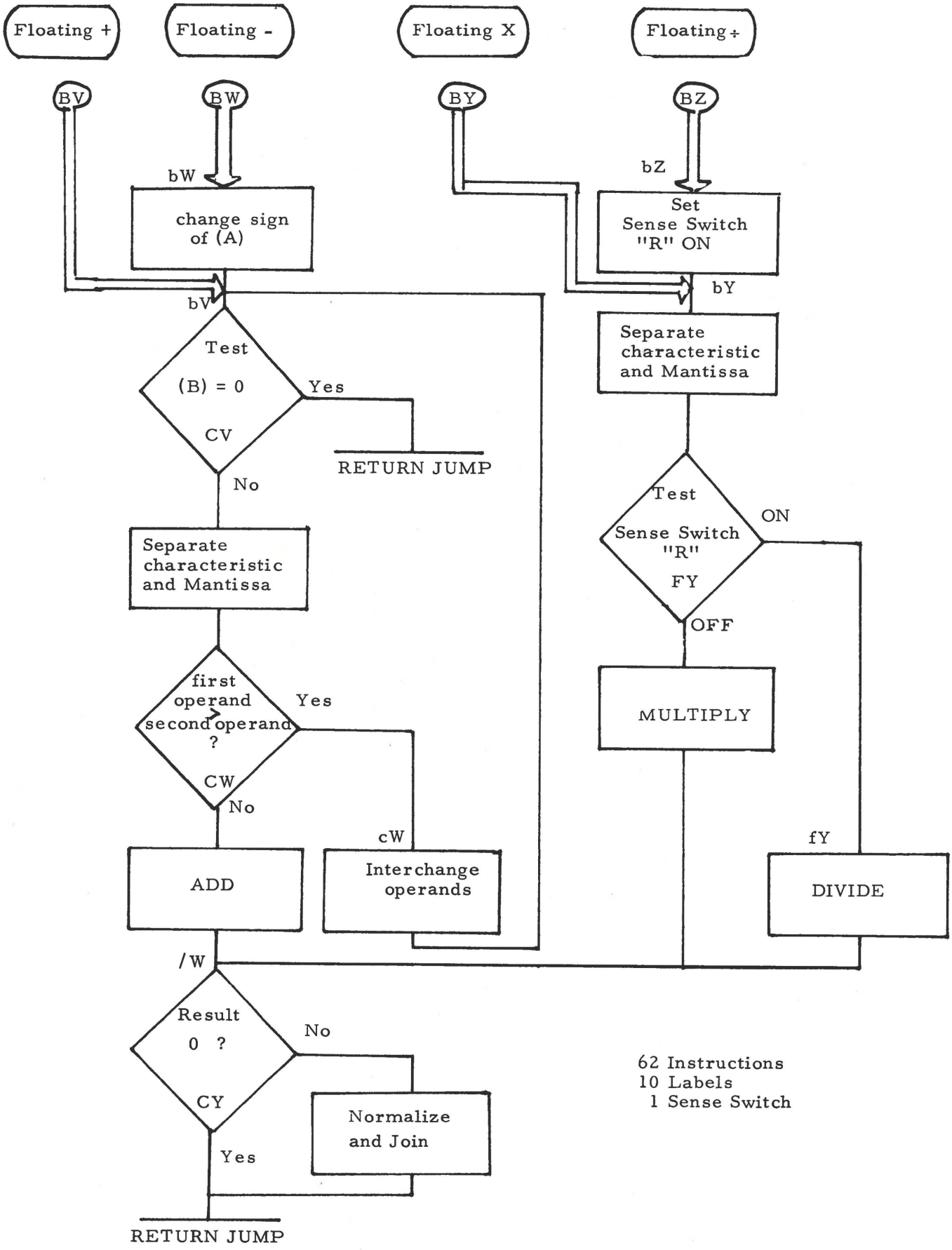
THE RESULTS OF ALL ROM FLOATING POINT ROUTINES ARE ALWAYS LEFT NORMALIZED.

Complete Routine for Floating Point Operations:

1	bW
2	A↕
3	bV
4	b↕
5	B↓
6	RS
7	CV
8	r↕
9	R↑
10	b↓
11	r↕
12	↓
13	R-
14	AS
15	CW
16	b↕
17	r↕
18	b+
19	-
20	rX
21	B↕
22	r↕
23	b↑
24	B+
25	/W
26	RS
27	CY
28	r÷
29	b↕
30	+
31	↕
32	b↓

33	/↕
34	cY
35	R√
36	bZ
37	r+
38	bY
39	r↕
40	R↑
41	B↕
42	r↕
43	b↕
44	↕
45	rS
46	FY
47	R+
48	b↕
49	BX
50	W
51	fY
52	R-
53	b↕
54	B÷
55	W
56	cW
57	b↓
58	B↕
59	BV
60	cV
61	b↓
62	R√

This sequence corresponds to the following flow chart. The first instruction here corresponds to instruction 1 in the complete ROM chart.



62 Instructions
10 Labels
1 Sense Switch

olivetti P 602

Code	20.5	Date		Page	1/1

Title
Complete ROM

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

1	bW	65	aZ	129	+	193	B↓	257	R◊	321	BX	385	A↑	449	R+
2	A↕	66	A↑	130	BX	194	-	258	RX	322	B↑	386	d↓	450	R↑
3	bV	67	R↕	131	A↑	195	-	259	R↑	323	BX	387	-	451	dS
4	b↕	68	R*	132	R◊	196	B+	260	R↕	324	B◊	388	b↑	452	+
5	B↓	69	RS	133	R+	197	AX	261	RX	325	FS	389	B↑	453	BX
6	RS	70	R-	134	R÷	198	B↑	262	R*	326	b↑	390	AS	454	A↑
7	CV	71	R◊	135	R+	199	A↑	263	R÷	327	+	391	DY	455	B-
8	r↕	72	R-	136	R÷	200	R↓	264	RS	328	BX	392	r+	456	B-
9	R↑	73	R↑	137	RX	201	R*	265	B÷	329	A↑	393	+	457	B↑
10	b↓	74	RS	138	R*	202	R↑	266	R-	330	R-	394	eZ	458	B↑
11	r↕	75	R↕	139	R+	203	RS	267	d↓	331	R↑	395	b↕	459	B*
12	↓	76	d↑	140	dS	204	d↓	268	aS	332	R↑	396	A+	460	B◊
13	R-	77	aS	141	+	205	↕	269	CY	333	R↕	397	A✓	461	BX
14	AS	78	CY	142	BX	206	X	270	a+	334	R*	398	bX	462	B↕
15	CW	79	rS	143	A↑	207	A↑	271	rS	335	RX	399	b↕	463	B↑
16	b↕	80	EZ	144	R-	208	R◊	272	FV	336	RX	400	B-	464	fS
17	r↕	81	÷	145	R+	209	R↓	273	-	337	R↕	401	↕	465	+
18	b+	82	/Z	146	R*	210	R*	274	fV	338	D-	402	b↓	466	BX
19	-	83	r↕	147	RX	211	R◊	275	+	339	r↑	403	-	467	A↑
20	aX	84	b↑	148	R↑	212	R↕	276	A↑	340	+	404	b↕	468	R◊
21	b↕	85	A↑	149	R-	213	d↓	277	d+	341	BX	405	+	469	RS
22	r↕	86	f↓	150	R-	214	+	278	÷	342	A↑	406	b+	470	RX
23	b↑	87	+	151	R↕	215	BX	279	R↓	343	B↕	407	AX	471	R-
24	B+	88	b↕	152	RS	216	X	280	A↓	344	B+	408	b↑	472	R↕
25	/W	89	-	153	d↓	217	A↑	281	A↑	345	B↕	409	A↑	473	R↓
26	RS	90	b↕	154	+	218	R+	282	d↓	346	B-	410	f↓	474	R↑
27	CY	91	r↕	155	BX	219	R+	283	-	347	BS	411	aX	475	R+
28	a÷	92	B↕	156	A↑	220	RS	284	-	348	BS	412	B↕	476	R↓
29	b↕	93	b↓	157	R+	221	R+	285	A↓	349	B÷	413	A↑	477	R+
30	+	94	+	158	R↕	222	R↑	286	-	350	B↕	414	B↓	478	dS
31	↕	95	b↕	159	R◊	223	RX	287	AS	351	B↕	415	B*	479	+
32	b↓	96	B↓	160	RX	224	R↓	288	FW	352	F↑	416	B-	480	bX
33	/↕	97	A↑	161	R-	225	R↕	289	a-	353	b↓	417	B↕	481	A↑
34	cY	98	f↓	162	R↓	226	d↑	290	fW	354	+	418	fS	482	R+
35	R✓	99	aX	163	R↓	227	+	291	A↓	355	BX	419	↓	483	R↓
36	bZ	100	B↕	164	R*	228	BX	292	-	356	A↑	420	BX	484	R◊
37	r+	101	A↑	165	R↕	229	X	293	AX	357	d↓	421	A↑	485	RS
38	bY	102	R-	166	R+	230	A↑	294	A↑	358	+	422	R+	486	R*
39	r↕	103	R↕	167	d↓	231	R-	295	f↓	359	aS	423	R:	487	R*
40	R↑	104	R↑	168	+	232	R-	296	aX	360	EV	424	RS	488	RX
41	B↕	105	R↓	169	BX	233	RS	297	B↕	361	A↕	425	RX	489	R↑
42	r↕	106	RS	170	A↑	234	R+	298	A↑	362	eV	426	R+	490	R*
43	b↕	107	dS	171	d↓	235	R↓	299	B-	363	a-	427	R↕	491	R↕
44	↕	108	↓	172	+	236	R↓	300	B+	364	/S	428	dS	492	dS
45	rS	109	BX	173	AX	237	R÷	301	B↑	365	EW	429	+	493	+
46	FY	110	A↑	174	AX	238	R◊	302	B◊	366	R✓	430	BX	494	rS
47	R+	111	R+	175	AX	239	R+	303	B↕	367	aW	431	A↑	495	DW
48	b↕	112	R◊	176	AX	240	R*	304	f↑	368	a+	432	B*	496	B↕
49	BX	113	R+	177	b+	241	dX	305	↓	369	RV	433	B↕	497	↓
50	W	114	R↓	178	-	242	+	306	BX	370	eW	434	BX	498	A+
51	fY	115	R-	179	R✓	243	BX	307	A↑	371	rS	435	B-	499	A+
52	R-	116	RS	180	fZ	244	b+	308	R*	372	EY	436	B:	500	B-
53	b↕	117	dS	181	X	245	rS	309	R↕	373	b÷	437	B*	501	dW
54	B÷	118	+	182	R✓	246	CY	310	R↑	374	R✓	438	B↑	502	/S
55	W	119	BX	183	/Y	247	r+	311	R÷	375	eY	439	fS	503	DV
56	cW	120	A↑	184	r+	248	AZ	312	R÷	376	r-	440	+	504	R✓
57	b↓	121	R↕	185	aY	249	rY	313	R↓	377	b↕	441	BX	505	dV
58	b↕	122	R*	186	a+	250	/+	314	d*	378	RV	442	A↑	506	A↕
59	BV	123	R↓	187	b↑	251	b↕	315	+	379	rZ	443	R◊	507	R✓
60	cV	124	R+	188	A✓	252	b↓	316	BX	380	AS	444	R↕	508	dY
61	b↓	125	R◊	189	A✓	253	rW	317	A↑	381	DZ	445	R*	509	+
62	R✓	126	R+	190	A✓	254	r+	318	B-	382	A↓	446	R↑	510	↕
63	aV	127	R↓	191	+	255	rV	319	B+	383	/+	447	R◊	511	÷
64	a+	128	dS	192	B↕	256	A↑	320	B◊	384	dZ	448	R↕	512	EZ

Real-Valued Functions

Title		
sinh x		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$\sinh x = \frac{e^{x} - e^{-x}}{2} = -\tfrac{1}{2}\left[\frac{1}{e^{x}} - e^{x}\right]$$

SPECIFICATION(S) OR ARGUMENT(S):

$|x| < 34$

LOGIC AND STORAGE:

1 GENERATED CONSTANT: -2
2 WORKING REGISTERS: b, B - half registers

ERROR:

There is an error in the tenth significant digit.

RUNNING TIME:

6 seconds

ARGUMENT(S):

x in A

FUNCTION(S) OR RESULT(S):

sinh x is left in A

PROGRAM LISTING:

```
/ *
AZ
* X
 ↕
 +
 -
A↑
f↑
 +
```

SAMPLE RUN:

```
                                   V
x                          1       S
sinh x          1.1752011937       A◊

x                         -1       S
sinh x         -1.1752011936       A◊

                3.1415926535       S
               11.5487393571       A◊
```

Title		
cosh x		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{2}\left[e^x + \frac{1}{e^x} \right]$$

SPECIFICATION(S) ON ARGUMENT(S):

$|x| < 34$

LOGIC AND STORAGE:

1 GENERATED CONSTANT: 2
2 WORKING REGISTERS: b, B - half registers

ERROR:

There is an error in the tenth significant digit.

RUNNING TIME:

6 seconds

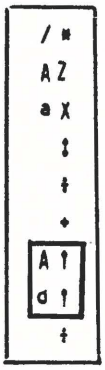
ARGUMENT(S):

x in A

FUNCTION(S) OR RESULT(S):

cosh x is left in A

PROGRAM LISTING:



SAMPLE RUN:

		V
x	1	S
cosh x	1.5430806349	A 0
x	-1	S
cosh x	1.5430806348	A 0
x	3.1415926535	S
cosh x	11.5919532754	A 0

Title		
tanh x		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} = 1 - \frac{2}{e^{2x} + 1}$$

The calculations are carried out in floating point to avoid overflow. The answer is left in fixed point.

LOGIC AND STORAGE:

2 GENERATED CONSTANTS: 1.1, 1.-2
2 WORKING REGISTERS: b, B - half registers

ERROR:

There is an error in the tenth decimal place

RUNNING TIME:

10 seconds

ARGUMENT(S):

x in A

FUNCTION(S) OR RESULT(S):

tanh x is left in A

PROGRAM LISTING:

```
A ÷
/ ×
AZ
/ ×
 W
A ↑
R ↓
d ↓
B ↑
/ ×
BV
A ↑
B ↑
d ↓
B ↑
/ ×
BZ
```

SAMPLE RUN:

```
                                   V
x                          0.5     S
tanh x          0.4621171572       A0

x                            6     S
tanh x          0.9999877116       A0

                           0.6     S
                0.5370495670       A0

                           0.7     S
                0.6043677771       A0

                           0.8     S
                0.6640367702       A0
```

Title		
arcsin x		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$\arcsin x = 2 \arctan \left[\frac{x}{1 + \sqrt{1 - x^2}} \right]$$

SPECIFICATION(S) ON ARGUMENT(S):

$|x| \leq 1$

LOGIC AND STORAGE:

1 GENERATED CONSTANT: -1
2 WORKING REGISTERS: b, B - half registers

ERROR:

There is an error in the tenth decimal place.

RUNNING TIME:

7 seconds

ARGUMENT(S):

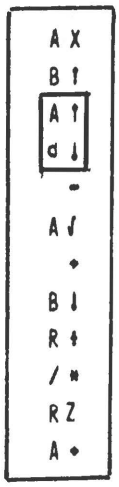
x in A

FUNCTION(S) OR RESULT(S):

arcsin x is left in A

$$-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}$$

PROGRAM LISTING:



SAMPLE RUN:

```
                                        V
x                              0.5      S
arcsin x          0.5235987753          A◊

x                              0.111    S
arcsin x          0.1112292117          A◊

                               0        S
                 -0.0000000002          A◊

                              -1        S
                 -1.5707963266          A◊

                               1        S
                  1.5707963266          A◊
```

Title		
arccos x		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$\arccos x = \frac{\pi}{2} - \arcsin x = \frac{\pi}{2} - 2 \arctan \left(\frac{x}{1 + \sqrt{1 - x^2}} \right)$$

SPECIFICATION(S) ON ARGUMENT(S):

$|x| \leq 1$

LOGIC AND STORAGE:

2 GENERATED CONSTANTS: 1, $\pi/2$
2 WORKING REGISTERS: b, B - half registers

ERROR:

There is an error in the tenth decimal place

RUNNING TIME:

7 seconds

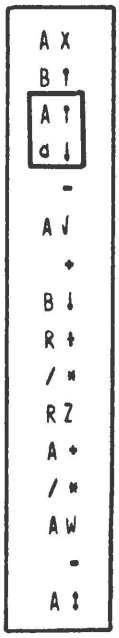
ARGUMENT(S):

x in A

FUNCTION(S) OR REGULT(S):

arccos x is left in A
$0 \leq \arccos x \leq \pi$

PROGRAM LISTING:



SAMPLE RUN:

```
                                        V
x                          0.577        S
arccos x          0.9557455433          A◊

x                         -0.315        S
arccos x          1.8912529810          A◊

                          -0.9999       S
                  3.1274504003          A◊

                             -1         S
                  3.1415926534          A◊

                              0         S
                  1.5707963270          A◊

                              1         S
                  0.0000000002          A◊
```

Title
arctanh x

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$\text{arctanh } x = \frac{1}{2} \ln \left(\frac{1 + x}{1 - x} \right)$$

Note: $\text{arcsinh } x = \text{arctanh} \left[\frac{x}{\sqrt{1 + x^2}} \right]$

$\text{arccosh } x = \text{arctanh} \left[\frac{\sqrt{x^2 - 1}}{x} \right]$

SPECIFICATION(S) ON ARGUMENT(S):

$|x| < 1$

LOGIC AND STORAGE:

2 GENERATED CONSTANTS: 1, 2
2 WORKING REGISTERS: b, B - half registers

ERROR:

There is an error in the tenth decimal place.

RUNNING TIME:

6 seconds

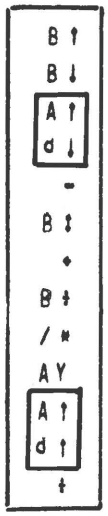
ARGUMENT(S):

x in B

FUNCTION(S) OR RESULT(S):

arctanh x is left in A

PROGRAM LISTING:



SAMPLE RUN:

```
                                        V
x                              0.5      S
arctanh x          0.5493061442         A 0

x                              0.6      S
arctanh x          0.6931471805         A 0

                               0.7      S
                   0.8673005277         A 0

                               0.9      S
                   1.4722194896         A 0

                           0.99999      S
                   6.1030338228         A 0
```

Title		
Rectangular to Polar Coordinates		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

For $Y \neq 0$, $\begin{cases} R = \sqrt{X^2 + Y^2} \\ \theta = 2 \arctan \left[\frac{R - X}{Y} \right] \end{cases}$

For $Y = 0$, $X \neq 0$, $\begin{cases} R = |X| \\ \theta = \frac{\pi}{2} \left(1 - \frac{X}{|X|} \right) \end{cases}$

For $X = Y = 0$, $R = \theta = 0$

SPECIFICATION(S) ON ARGUMENT(S):

X and Y arbitrary

LOGIC AND STORAGE:

2 LABELS: fV, fW
3 WORKING REGISTERS: b, B, c - half registers

ERROR:

R is accurate to the Decimal Wheel Setting. θ has an error in the 10th decimal place.

RUNNING TIME:

6 seconds

ARGUMENT(S)

X in C/
Y in A

FUNCTION(S) OR RESULT(S):

R in C/
θ in A, $-\pi < \theta \leq \pi$

PROGRAM LISTING:

```
R S
F V
A X
b ↑
B ↕
c ↓
A X
B •
A /
c ↕
c -
A ↕
b +
/ *
R Z
A •
F W
f V
c ↓
R S
F W
A ↓
c ↕
c +
/ *
A W
  X
A ↕
  •
f W
```

SAMPLE RUN:

```
                                        V
x                    123.456            S
y                   -654.321            S
R          665.8658686079                c◊
θ           -1.3843105948                A◊

x                   -951.235            S
y                          0            S
R          951.2350000000                c◊
θ            3.1415926536                A◊

                           0            S
                           0            S
           0.0000000000                  c◊
           0.0000000000                  A◊
```

9/19/80

STORE x in C/

Y in A

KEY V

Read C/ = R

Read A = θ = AΣ in radians

Title		
Polar to Rectangular Coordinates		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$X = R \cos \theta$
$Y = R \sin \theta$

SPECIFICATION(S) ON ARGUMENT(S):

θ must be given in radians.

LOGIC AND STORAGE:

2 WORKING REGISTERS: b, B - half registers

ERROR:

There is an error in the tenth decimal place in the sin and cos.

RUNNING TIME:

8 seconds

ARGUMENT(S):

θ in B
R in C

FUNCTION(S) OR RESULT(S):

X is left in A
Y is left in b

PROGRAM LISTING:

```
D↓
/*
RV
CX
D↕
/*
RW
CX
```

SAMPLE RUN:

```
                                   V
θ                          25      S
R                          0.5     S
x          21.9395640567           A◊
y          11.9856384755           D◊

θ                          36      S
R                          -2      S
x          -14.9812861309          A◊
y          -32.7347073814          D◊

                           14      S
                  3.1415926535     S
          -14.0000000000           A◊
           0.0000000028            D◊

                          156      S
                         1.2356    S
          51.3169139881            A◊
         147.3179363196            D◊
```

Complex Valued Functions

(Real and Imaginary Parts)

Title		
e^Z (Z complex)		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$W = e^{(x + iy)} = e^x(\cos y + i \sin y)$$

SPECIFICATION(SO ON ARGUMENT(S):

x very large will result in overflow

LOGIC AND STORAGE:

3 WORKING REGISTERS: B, b, C - half registers

ERROR:

There is an error in the 10th significa nt digit.

RUNNING TIME:

13 seconds

ARGUMENT(S):

x in A
y in C

FUNCTION(S) OR RESULT(S):

Re [W] in A
Im [W] in c

PROGRAM LISTING:

```
/*
AZ
a X
b :
c ;
/*
RV
b X
c :
/*
RW
b X
```

SAMPLE RUN:

$$W = e^{(x + iy)}$$

```
                                    V

            x =             1       S
            y =             0       S

Re[W] =       2.7182818 28          A 0
Im [W] =     -0.0000000 00          c 0

                                    0       S
              3.1415926536          S

             -1.0000000 00          A 0
             -0.0000000 00          c 0

                                    2       S
              0.5235987756          S

              6.3991102 93          A 0
              3.6945280 52          c 0
```

Title		
Z^a (Z complex, a real)		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$W = Z^a = R^a (\cos a\theta + i \sin a\theta)$$

if $y \neq 0$; $W = (x^2+y^2)^{a/2} \left(\cos \left[2\text{atan}^{-1} \left(\frac{(x^2+y^2)^{\frac{1}{2}} - X}{y} \right) \right] + i \sin \left[2\text{atan}^{-1} \left(\frac{(x^2+y^2)^{\frac{1}{2}} - x}{y} \right) \right] \right)$

if $y = 0$; $\theta = \frac{\pi}{2} \left(1 - \frac{|x|}{x}\right)$

$$W = |x|^a (\cos a\theta + i \sin a\theta)$$

LOGIC AND STORAGE:

2 LABELS: fY, fw
4 WORKING REGISTERS: b, B, c, C = half registers

ERROR:

Maximum accuracy is 9 significant digits. For x much larger than y, the error increases.

RUNNING TIME:

25 seconds

ARGUMENT(S):

x in C
y in A
a in C

FUNCTION(S) OR RESULT(S):

Re [W] in A
Im [W] in c

PROGRAM LISTING:

```
RS
FY
AX
B↑
c↓
b↑
AX
B*
A√
c↕
c-
A↕
b↕
/*
RZ
A*
FW
fY
c↓
RS
FW
A↓
c↕
/*
AW
 X
A↕
 *
fW
CX
c↕
/*
```

```
 Y
CX
/*
 Z
*X
C↕
c↓
/*
RV
CX
c↕
/*
RW
CX
```

SAMPLE RUN:

$$W = (x + iy)^a$$

```
                                        V

                x =          25         S
                y =          25         S
                a =         0.5         S

Re [W] =   5.4934205691                 A◊
Im [W] =   2.2754493047                 c◊

                             1          S
                             1          S
                           3.5          S

                -3.1075479500           A◊
                 1.2871885073           c◊

                             1          S
                             1          S
                           0.7          S

                 1.0867415831           A◊
                 0.6659560988           c◊

                 1.0867415831           S
                 0.6659560988           S
                            5           S

                -3.1075479535           A◊
                 1.2871885094           c◊
```

Title		
Z^u (Z, u complex)		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$Z = x + iy \quad , \quad u = a + ib$$

if $y \neq 0$; $\quad R = \sqrt{x^2 + y^2} \quad , \quad \theta = 2 \tan^{-1} \left(\frac{R - x}{y} \right)$

if $y = 0, x \neq 0$; $\quad R = |x| \quad , \quad \theta = \frac{\pi}{2} \left(1 - \frac{x}{x} \right)$

if $y = x = 0$; $\quad R = \theta = 0$

$$W = Z^u = e^{u \ln Z} = e^{u (\ln R + i\theta)}$$

$$= e^{(a + bi)(\ln R + i\theta)} = e^{a \ln R - b\theta + i(a\theta + b \ln R)}$$

$$= e^{a \ln R - b\theta} \left[\cos (a\theta + b \ln R) + i \sin (a\theta + b \ln R) \right]$$

LOGIC AND STORAGE:

2 LABELS: fy, fw
5 WORKING REGISTERS: b, B, c, C, d - half registers

ERROR:

Maximum accuracy is
9 significant digits.

RUNNING TIME:

25 seconds

ARGUMENT(S):

x in d
y in b
a in c
b in c

FUNCTION(S) OR RESULT(S):

Re [W] in A
Im [W] in c

PROGRAM LISTING:

```
b ↓
R S
F Y
A X
B ↕
d ↓
A X
B •
A ∫
d ↕
d -
b ↑
A ↕
/ *
R Z
A •
F W
f Y
d ↓
R S
F W
A ↓
d ↕
d ↑
/ *
A W
  X
A ↕
  •
f W
d ↕
/ *
```

```
A Y
b ↕
d ↓
c X
d ↕
C X
C ↕
b X
d -
c ↕
b X
C •
c ↕
/ *
A Z
a X
C ↕
c ↓
/ *
R V
C X
c ↕
/ *
R W
C X
```

SAMPLE RUN:

$$W = (x + iy)^{a + ib}$$

```
                                        V

              x =          25           S
              y =          25           S

              a =         0.5           S
              b =           0           S

Re [W] =  5.4934205691                  A◊
Im [W] =  2.2754493047                  c◊

                        1.23            S
                        4.56            S

                         0.1            S
                         0.3            S

          0.6527828632                  A◊
          0.4432130156                  c◊

          0.6527828632                  S
          0.4432130156                  S

                           1            S
                          -3            S

          1.2300000050                  A◊
          4.5600000023                  c◊

          2.7182818285                  S
                       0                S

                       0                S
          1.5707963268                  S

          0.0000000002                  A◊
          1.0000000000                  c◊
```

Title		
Ln (Z) (Z complex)		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$W = \ln(Z) = \ln(R) + i\theta$$

For $y \neq 0$; $\quad R = \sqrt{x^2 + y^2}$

$$\theta = 2 \tan^{-1} \left[\frac{R - x}{y} \right]$$

For $y = 0$; $\quad R = |x|$

$$\theta = \frac{\pi}{2} \left(1 - \frac{x}{X}\right)$$

SPECIFICATION(S) ON ARGUMENT(S):

$Z \neq 0$

LOGIC AND STORAGE:

2 LABELS: fY, eY
3 WORKING REGISTERS: b, B, c - half registers

ERROR:

There is an error in the 10th significant digit.

RUNNING TIME:

12 seconds

ARGUMENT(S):

x in c
y in A

FUNCTION(S) OR RESULT(S):

Re [W] in A
Im [W] in c $\quad -\pi < \theta \leq \pi$

PROGRAM LISTING:

```
RS
FY
AX
B↕
c↓
b↑
AX
B•
A√
c↕
c-
A↕
b+
/*
RZ
A•
c↕
/*
AY
EY
fY
c↓
A↓
c+
/*
AW
 X
A↕
 •
c↕
/*
AY
eY
```

SAMPLE RUN:

$$W = \ln (x + iy)$$

```
                                        V

                    x =     1           S
                    y =     0           S

Re [W] =    0.000000000     A◊
Im [W] =   -0.000000000     c◊

                   -1           S
                    0           S

            0.000000000     A◊
            3.141592653     c◊

            6.39911029          S
            3.69452805          S

            2.000000000     A◊
            0.523598775     c◊
```

Title
sin (Z) (Z complex)

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$W = \text{Sin } Z = \sin(x + iy) = \sin(x)\cos(iy) + \sin(iy)\cos(x)$$

$$= \sin(x)\cosh(y) + i\sinh(y)\cos(x)$$

$$= \sin(x)\left[\frac{e^{y} + e^{-y}}{2}\right] + i\cos(x)\left[\frac{e^{y} - e^{-y}}{2}\right]$$

SPECIFICATION(S) ON ARGUMENT(S):

$|y|$ very large will result in overflow

LOGIC AND STORAGE:

3 WORKING REGISTERS: B, b,c - half registers
1 CONSTANT REGISTER: (F = 2)

ERROR:

There is an error in the 10th significant digit.

RUNNING TIME:

14 seconds

ARGUMENT(S):

x in C
y in A

FUNCTION(S) OR RESULT(S):

Re [W] in A
Im [W] in B

PROGRAM LISTING:

```
/ *
A Z
a ×
b ↕
c ↓
/ *
R V
c ↕
/ *
R W
B ↕
  ↓
b +
  -
F +
B ↕
B ×
A ↕
B ↕
b ÷
c ×
```

2 F ◊

SAMPLE RUN:

		V
x =	0	S
y =	0	S
Re [W] =	-0.000000000	A ◊
Im [W] =	-0.000000000	B ◊
	0.5235987756	S
	0	S
	0.500000000	A ◊
	-0.000000000	B ◊
	0.5235987756	S
	1	S
	0.771540318	A ◊
	1.017754088	B ◊
	0	S
	1	S
	-0.000000000	A ◊
	1.175201193	B ◊

Title
cos (Z) (Z complex)

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$W = \cos Z = \cos (x + iy) = \cos (x) \cos (iy) - \sin (x) \sin (iy)$$

$$= \cos (x) \cosh (y) - i \sin (x) \sinh (y)$$

$$= \cos (x) \left[\frac{e^{y} + e^{-y}}{2} \right] + i \sin (x) \left[\frac{e^{-y} - e^{y}}{2} \right]$$

SPECIFICATION(S) ON ARGUMENT(S):

$|y|$ very large will result in overflow.

LOGIC AND STORAGE:

3 WORKING REGISTERS: B, b, c - half registers
1 CONSTANT REGISTER: (F = 2)

ERROR:

There is an error in the 10th significant digit.

RUNNING TIME:

14 seconds

ARGUMENT(S):

x in C
y in A

FUNCTION(S) OR RESULT(S):

Re [W] in A
Im [W] in B

PROGRAM LISTING:

```
/ *
AZ
* X
b ↕
c ↓
/ *
RW
c ↕
/ *
RV
B ↕
  ↓
b ↕
  -
F ↕
B ↕
B X
B ↕
b *
c X
```

2 F ◊

SAMPLE RUN:

```
W = cos (x + iy)                                  V

                                  x =    0        S
                                  y =    0        S

Re[W] =  1.000000000                              A◊
Im[W] = -0.000000000                              B◊

                          0.5235987756            S
                                     0            S

                          0.866025404             A◊
                          0.000000000             B◊

                          0.5235987756            S
                                     1            S

                          1.336347030             A◊
                         -0.587600597             B◊

                                     0            S
                                     1            S

                          1.543080634             A◊
                          0.000000000             B◊
```

Title		
sinh (Z) (Z complex)		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$W = \sinh(Z) = \frac{e^{Z} - e^{-Z}}{2}$$

$$= \frac{e^{x}(\cos y + i \sin y) - e^{-x}(\cos(-y) + i \sin(-y))}{2}$$

$$= \cos(y)\left[\frac{e^{x} - e^{-x}}{2}\right] + i \sin(y)\left[\frac{e^{x} + e^{-x}}{2}\right]$$

SPECIFICATION(S) ON ARGUMENT(S):

x very large will result in overflow.

LOGIC AND STORAGE:

3 WORKING REGISTERS: B, b, c - half registers
1 CONSTANT REGISTER: (F = 2)

ERROR:

There is an error in the 10th significant digit.

RUNNING TIME:

14 seconds

ARGUMENT(S):

x in A
y in c

FUNCTION(S) OR RESULT(S):

Re [W] in A
Im [W] in b

PROGRAM LISTING:

```
/ *
AZ
a X
b ↕
c ↓
/ *
RV
c ↕
/ *
RW
B ↕
  ↓
b ↑
  -
F ↑
b ↕
b •
c X
b ↕
B X
A ↕
```

SAMPLE RUN:

```
W = sin (x + iy)
                                        V

                          x =     0     S
                          y =     0     S

Re [W] =    -0.000000000    A◊
Im [W] =    -0.000000000    b◊

                            1     S
                            0     S

                1.175201193     A◊
               -0.000000000     b◊

                            0     S
            1.5707963268        S

                0.000000000     A◊
                1.000000000     b◊

                            0     S
            3.14159265359       S

                0.000000000     A◊
                0.000000000     b◊

                            1     S
                            1     S

                0.634963915     A◊
                1.298457581     b◊
```

Title		
cosh (Z) (Z complex)		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$W = \cosh(Z) = \frac{e^{Z} + e^{-Z}}{2}$$

$$= \frac{e^{x}(\cos y + i \sin y) + e^{-x}(\cos(-y) + i \sin(-y))}{2}$$

$$= \cos(y)\left[\frac{e^{x} + e^{-x}}{2}\right] + i \sin(y)\left[\frac{e^{x} - e^{-x}}{2}\right]$$

SPECIFICATION(S) ON ARGUMENT(S):

$|x|$ very large will result in overflow.

LOGIC AND STORAGE:

3 WORKING REGISTERS: B, b, c - half registers
1 CONSTANT REGISTER: (F = 2)

ERROR:

There is an error in the 10th significant digit.

RUNNING TIME:

14 seconds

ARGUMENT(S):

x in A
y in c

FUNCTION(S) OR RESULT(S):

Re [W] in b
Im [W] in A

PROGRAM LISTING:

```
 / *
 A Z
 • X
 b ↕
 c ↓
 / *
 R V
 c ↕
 / *
 R W
 B ↕
   ↓
 D +
   -
 F +
 b ↕
 D ◊
 B X
 D ↕
 c X
 A ↕
```

SAMPLE RUN:

W = cosh (x + iy)

		V
x =	0	S
y =	0	S
Re [W] =	1.000000000	b ◊
Im [W] =	0.000000000	A ◊
	1	S
	0	S
	1.543080634	b ◊
	-0.000000000	A ◊
	0	S
	1.5707963268	S
	-0.000000000	b ◊
	-0.000000000	A ◊
	0	S
	3.14159265359	S
	-1.000000000	b ◊
	-0.000000000	A ◊
	1	S
	1	S
	0.833730025	b ◊
	0.988897706	A ◊

Title		
tan (Z) (Z complex)		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$W = \tan Z = \frac{\sin Z}{\cos Z} = \frac{2 \sin 2x + i\,(e^{2y} - e^{-2y})}{2 \cos 2x + (e^{2y} + e^{-2y})}\ ,\ \text{Where } X = (x + iy)$$

SPECIFICATION(S) ON ARGUMENT(S):

$|y|$ very large will result in overflow in intermediate steps
$\tan (x + iy) \to i$ as $|y| \to \infty$

LOGIC AND STORAGE:

3 WORKING REGISTERS: B, b, c - half registers
1 CONSTANT REGISTER: (F) = 2

ERROR:

There is an error in the 10th significant digit.

RUNNING TIME:

14 seconds

ARGUMENT(S):

x in c
y in A

FUNCTION(S) OR RESULT(S):

Re[W] in b
Im [W] in A

PROGRAM LISTING:

```
A *
/ *
AZ
a X
b ↕
c ↓
A *
/ *
RV
c ↕
A *
/ *
RW
B ↕
b ↓
  ↕
  ↑
b -
F ↑
b ↕
b *
B *
c ↕
c ↑
b ↕
c ↑
A ↕
```

2 F ◊

SAMPLE RUN:

$W = \tan(x + iy)$

```
                                   V

                x =     0          S
                y =     0          S

Re[W] = -0.000000000               b ◊
Im[W] = -0.000000000               A ◊

          0.5235987756             S
                   0               S

          0.577350269              b ◊
         -0.000000000              A ◊

          0.5235987756             S
                   1               S

          0.203187621              b ◊
          0.850937092              A ◊

                   0               S
                   1               S

         -0.000000000              b ◊
          0.761594155              A ◊

          0.5235987756             S
                  10               S

          0.000000003              b ◊
          0.999999997              A ◊
```

Title
$\tan^{-1}(Z)$ (Z complex)

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$W = \tan^{-1} Z \quad \text{where} \quad Z = \tan W = \frac{e^{iw} - e^{-iw}}{i(e^{iw} + e^{-iw})} \quad \text{and} \quad e^{2iw} = \frac{1 + Zi}{1 - Zi}$$

$$W = \frac{i}{2} \ln \left(\frac{1 - y^2 - x^2 - 2xi}{(1-y)^2 + x^2} \right)$$

if $X = 0$:

$$\text{Re}[W] = \frac{i^2}{2} \frac{\pi}{2} \left(1 - \frac{(1-y)^2}{|1-y^2|} \cdot \frac{1 - y^2}{(1-y)^2} \right)$$

since

$$-\frac{\pi}{2} < \text{Re}[W] \leq \frac{\pi}{2};$$

$$\text{Re}[W] \begin{cases} = \frac{\pi}{2} \text{ if } |y| > 1 \\ = 0 \text{ otherwise} \end{cases}$$

$$\text{Im}[W] = \frac{i}{2} \ln \left| \frac{1 - y^2}{(1-y)^2} \right| = \frac{i}{2} \ln \left| \frac{1 + y}{1 - y} \right|$$

if $X \neq 0$:

$$\text{Re}[W] = \frac{i \cdot i}{2} 2 \tan^{-1} \left[\frac{\sqrt{(1-y^2-x^2)^2 + yx^2} - 1 + x^2 + y^2}{-2x} \right]$$

$$= \tan^{-1} \left[\frac{\sqrt{x^2 + (y+1)^2}\ \sqrt{x^2 + (y-1)^2} - 1 + x^2 + y^2}{2x} \right]$$

$$\text{Im}[W] = \frac{i}{2} \ln \frac{\sqrt{(1-y^2-x^2)^2 + yx^2}}{(1-y)^2 + x^2} = \frac{i}{2} \ln \sqrt{\frac{(1+y)^2 + x^2}{(1-y)^2 + x^2}}$$

SPECIFICATION(S) ON ARGUMENT(S):

$Z \neq i$

LOGIC AND STORAGE:

2 LABELS:	fy, fw
4 WORKING REGISTERS:	B, b, C, c - half registers
2 CONSTANT REGISTERS:	(F) = 2, (f) = 1

ERROR:

There is an error in the 10th significant digit.

RUNNING TIME:

15 seconds

ARGUMENT(S):

x in C

y in B

FUNCTION(S) OR RESULT(S):

Re[W] in C

Im[W] in A

PROGRAM LISTING:

```
f ↓
B ◆
c ↕
f ↓
  −
C ↕
R S
F Y
A X
b ↑
C ↕
A X
C ◆
A √
c ↕
A X
C ◆
A √
C ↕
B ↕
A X
B ◆
f ◆
B ↕
C ↓
c X
B ◆
b +
F +
/ *
R Z
c ↕
```

```
C +
F W
f Y
c ↕
C ↕
C +
A S
F W
/ *
A W
c ↑
f W
/ *
A Y
F +
```

```
 2   F ◊
−1   f ◊
```

SAMPLE RUN:

$W = \tan^{-1}(x + iy)$ V

x =	0	S
y =	0	S
Re[W] =	0.000000000	c ◊
Im[W] =	0.000000000	A ◊
	1	S
	0	S
	0.785398163	c ◊
	0.000000000	A ◊
	0.577350269	S
	0	S
	0.523598775	c ◊
	0.000000000	A ◊
	0.203187621	S
	0.850937092	S
	0.523598775	c ◊
	1.000000000	A ◊
	0	S
	0.761594155	S
	0.000000000	c ◊
	0.999999997	A ◊
	0	S
	10	S
	1.570796326	c ◊
	0.100335347	A ◊

Complex-Valued Functions

(Modulus and Phase)

Title		
Modulus and Phase of Ln (Z)		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$Re^{i\theta} = \ln(x + iy) = \ln(r) + i\theta$$

$$r = \begin{cases} |x| & \text{if } y = 0 \\ \sqrt{X^2 + Y^2} & \text{if } y \neq 0 \end{cases}$$

$$\varphi = \begin{cases} \frac{\pi}{2}\left(1 - \frac{x}{|x|}\right) & \text{if } y = 0 \\ 2\tan^{-1}\left(\frac{r-x}{y}\right) & \text{if } y \neq 0 \end{cases}$$

$$R = \begin{cases} |\ln(r)| & \text{if } \varphi = 0 \quad (y = 0) \\ \sqrt{[\ln(r)]^2 + \varphi^2} & \text{if } \varphi \neq 0 \quad (y \neq 0) \end{cases}$$

$$\Theta = \begin{cases} \frac{\pi}{2}\left(1 - \frac{\ln r}{|\ln r|}\right) & \text{if } \varphi = 0 \quad (y = 0) \\ 2\tan^{-1}\left[\frac{R - \ln r}{\varphi}\right] & \text{if } \varphi \neq 0 \quad (y \neq 0) \end{cases}$$

SPECIFICATION(S) ON ARGUMENT(S):

$Z \neq 0$

LOGIC AND STORAGE:

1 LABEL:	fy
3 WORKING REGISTERS:	b, B, c - half registers
1 SUBROUTINE:	F√

ERROR:

There is an error in the 10th significant digit

RUNNING TIME:

18 seconds

ARGUMENT(S):

x in c
y in A

FUNCTION(S) OR RESULT(S):

R in c
θ in A $\quad -\pi < \theta \leq \pi$

PROGRAM LISTING:

```
f √
R S
F Y
A X
B ↑
b ↕
c ↓
A X
b ◆
A √
c ↕
c −
B +
A ↕
/ *
R Z
A ◆
f √
f Y
c ↓
A ↓
c ↕
c +
/ *
A W
  X
A ↕
  ◆
f √
```

```
F √
c ↕
/ *
A Y
c ↕
F √
```

SAMPLE RUN:

```
                              V

   x               1          S
   y               0          S

R      0.000000000            c ◊
      -0.000000000            A ◊

                  -1          S
                   0          S

       3.141592653            c ◊
       1.570796326            A ◊

      -0.989992496            S
       0.141120008            S

       3.000000000            c ◊
       1.570796327            A ◊
```

Title		
Modulus and Phase of sin (Z)		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$Re^{i\theta} = \sin(x + iy) = \sin X \left(\frac{e^{y} + e^{-y}}{2}\right) + i \cos x \left[\frac{e^{y} - e^{-y}}{2}\right]$$

$$R = \begin{cases} |\sin X| & \text{if } y = 0 \\ \frac{1}{2}\sqrt{\sin^2 x\,(e^{y} + e^{-y})^2 + \cos^2 x\,(e^{y} - e^{-y})^2} & \text{if } y \neq 0 \end{cases}$$

$$\theta = \begin{cases} \frac{\pi}{2}\left(1 - \frac{\sin X}{|\sin X|}\right) & \text{if } y = 0 \\ 2\tan^{-1}\left[\frac{R - \frac{1}{2}\sin X\,(e^{y} + e^{-y})}{\frac{1}{2}\cos X\,(e^{y} - e^{-y})}\right] & \text{if } y \neq 0 \text{ and } x \neq \frac{\pi}{2} \pm k\pi,\ k = 0, 1, 2\ldots \end{cases}$$

SPECIFICATION(S) ON ARGUMENT(S):

$|y|$ very large will result in overflow.
This routine does not preserve accuracy for $x = \frac{\pi}{2} + k\pi (k = 0, 1, 2, \ldots)$
This is due to fact that the value for $\cos(\frac{\pi}{2} \neq k\pi)$ will not be exactly zero, for all decimal wheel settings, hence this condition cannot be easily tested in the arctangent routine.

LOGIC AND STORAGE:

2 LABELS; fy, fw
3 WORKING REGISTERS: b, B, c - half registers
1 CONSTANT REGISTER: (F)= 2

ERROR:

There is an error in the 10th significant digit.

RUNNING TIME:

20 seconds

ARGUMENT(S):

x in C
y in A

FUNCTION(S) OR RESULT(S):

R in c
θ in A $\quad -\pi < \theta \leq \pi$

PROGRAM LISTING:

```
R S
F Y
/ *
A Z
a X
b ↕
c ↓
/ *
R V
c ↕
/ *
R W
B ↕
  ↓
b +
  −
F +
B ↕
B X
B ↕
b ◆
c X
A X
b ↕
c ↑
B ↓
B X
b ◆
A √
c ↕
c −
B +
```

```
/ *
R Z
A ◆
F W
f Y
c ↓
R S
F W
/ *
R V
R ↕
c ↕
R ↓
c +
/ *
A W
  X
A ↕
  ◆
f W
```

```
2   F ◇
```

SAMPLE RUN:

```
                      V

   x          0       S
   y          0       S

R   0.000000000      c ◇
θ   0.000000000      A ◇

   0.5235987756       S
              0       S

    0.500000000      c ◇
   -0.000000000      A ◇

  -0.5235987756       S
              0       S

    0.500000000      c ◇
    3.141592653      A ◇

              0       S
              1       S

    1.175201193      c ◇
    1.570796327      A ◇
```

Title
Modulus and Phase of cos (Z)

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$Re^{i\theta} = \cos(x + iy) = \cos x \left(\frac{e^{y} + e^{-y}}{2} \right) - i \sin x \left(\frac{e^{y} - e^{-y}}{2} \right)$$

$$R = \begin{cases} |\cos x| & \text{if } y = 0 \\ \frac{1}{2}\sqrt{\sin^2 x \left(e^{y} - e^{-y} \right)^2 + \cos^2 \left(e^{y} + e^{-y} \right)^2} & \text{if } y \neq 0 \end{cases}$$

$$\theta = \begin{cases} \frac{\pi}{2} \left[1 - \frac{\cos x}{|\cos x|} \right] & \text{if } y = 0 \\ 2 \tan^{-1} \left[\frac{R - \frac{1}{2}\cos x \left(e^{y} + e^{-y} \right)}{\frac{1}{2} \sin x \left(e^{-y} - ey \right)} \right] & \text{if } y \neq 0 \text{ and } x \neq K\pi; \\ & (K = 0, 1, 2, \ldots) \end{cases}$$

SPECIFICATION(S) ON ARGUMENT(S):

|y| very large will result in overflow.
This subroutine does not preserve accuracy for $x = k\pi$, (k = 1, 2, . . .)
See comment in Subroutine 21.51

LOGIC AND STORAGE:

2 LABELS: fy, fw
3 WORKING REGISTERS: b, B, c - half registers
1 CONSTANT REGISTER: (F)= -2

ERROR:

There is an error in the 10th significant digit.

RUNNING TIME:

20 seconds

ARGUMENT(S):

x in c
y in A

FUNCTION(S) OR RESULT(S):

R in c
θ in A $\quad -\pi < \theta \leq \pi$

PROGRAM LISTING:

```
 R S        / *
 F Y        R Z
 / *        A ◆
 A Z        F W
 a X        f Y
 b ↕        c ↓
 c ↓        / *
 / *        R W
 R W        R ↕
 c ↕        c ↕
 / *        R ↓
 R V        c ‡
 B ↕        / *
   ↓        A W
 b ‡          X
   -        A ↕
 F ‡          ◆
 B ↕        f W
 B X
 A X        - 2   F ◊
 B ↕
   ↕
 b ↕
   -
 c X
 A X
 c ↑
 B ◆
 A √
 c ↕
 c -
 b ‡
```

SAMPLE RUN:

```
                          V

      x            0      S
      y            0      S

R        1.000000000      c ◊
θ       -0.000000000      A ◊

        0.5235987756      S
                   0      S

         0.866025404      c ◊
        -0.000000000      A ◊

        0.5235987756      S
                   1      S

         1.459828019      c ◊
        -0.414261029      A ◊
```

Title		
Modulus and Phase of tan (Z)		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$Re^{i\theta} = \tan(x + iy) = \frac{\sin 2x + i\left(\frac{e^{2y} - e^{-2y}}{2}\right)}{\cos 2x + \left(\frac{e^{2y} + e^{-2y}}{2}\right)}$$

$$R = \begin{cases} \tan x & \text{if } y = 0 \\ \sqrt{\frac{a}{b}} & \text{if } y \neq 0 \end{cases}$$

$$\theta = \begin{cases} \frac{\pi}{2}\left(1 - \frac{\tan x}{\tan x}\right) & \text{if } y = 0 \\ 2\tan^{-1}\left[\frac{ab - \sin 2x}{\frac{e^{2y} - e^{-2y}}{2}}\right] & \text{if } y \neq 0 \end{cases}$$

where $a = \frac{e^{2y} + e^{-2y}}{2} - \cos 2x$

$$b = \frac{e^{2y} + e^{-2y}}{2} + \cos 2x$$

SPECIFICATION(S) ON ARGUMENT(S):

$|y|$ very large will result in overflow in intermediate steps.
$\tan(x + iy) \rightarrow 1e^{\pi/2\theta}$ as $|y| \rightarrow \infty$

LOGIC AND STORAGE:

2 LABELS:	fy, fw
3 WORKING REGISTERS:	B, b, c, - half registers
1 CONSTANT REGISTER:	(F) = 2

ERROR:

There is an error in the 10th significant digit.

RUNNING TIME:

20 seconds

ARGUMENT(S):

X in C
y in A

FUNCTION(S) OR RESULT(S):

R in c
θ in A $-\pi < \theta \leq \pi$

PROGRAM LISTING:

```
R S      X
F Y      ↕
A •    c ↕
/ ✱      -
A Z    b ÷
a X    B ↕
b ↕    c ÷
c ↓    c ↕
A •    B ↓
/ ✱    / ✱
R V    R Z
c ↕    A •
A •    F W
/ ✱    f Y
R W    c ↓
B ↕    R S
  ↓    F W
b ÷    / ✱
  -    R Y
F ÷    c ↕
b ↕    c ↓
b •    A ↓
B ↕    c ↕
R ↕    c ÷
R •    / ✱
B ↕    A W
  -      X
A √    A ↕
B ↕      •
A √    f W
  ↕
B ↓
       2   F ◊
```

SAMPLE RUN:

$$Re^{i\theta} = \tan(x + iy)$$

		V
x	0	S
y	0	S
R	0.000000000	c ◊
θ	0.000000000	A ◊
	0.5235987756	S
	0	S
	0.577350269	c ◊
	-0.000000000	A ◊
	0	S
	1	S
	0.761594155	c ◊
	1.570796327	A ◊
	0.5235987756	S
	1	S
	0.874859499	c ◊
	1.336404261	A ◊
	0.5235987756	S
	10	S
	0.999999997	c ◊
	1.570796323	A ◊

Complex Number Manipulation

Title

$\frac{1}{Z}$ (Z complex)

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$W = \frac{1}{Z} = \frac{1}{x + iy} \quad \frac{x - iy}{x - iy} = \frac{x - iy}{x^2 + y^2}$$

SPECIFICATION(S) ON ARGUMENT(S):

$Z \neq 0$

LOGIC AND STORAGE:

2 WORKING REGISTERS: b, B - half registers

RUNNING TIME:

2 seconds

ARGUMENT(S):

x in B
y in A

FUNCTION(S) OR RESULT(S):

Re [W] in b
Im [W] in A

PROGRAM LISTING:

A ↕
A X
b ↑
B ↕
A X
B ↓
B ↑
R ÷
B ↕
B ‡
b ↕
‡

SAMPLE RUN:

		V
x =	0	S
y =	10	S
Re [W] =	0.000	b ◊
Im [W] =	-0.100	A ◊
	10	S
	0	S
	0.100	b ◊
	-0.000	A ◊
	3.75	S
	2.95	S
	0.16472655	b ◊
	-0.12958488	A ◊

Title		
$Z_1 Z_2$ (complex)		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$w = Z_1 Z_2 = (x_1 + iY_1)(x_2 + iY_2) = x_1x_2 - Y_1Y_2 + i\ (Y_1x_2 + Y_2x_1)$$

LOGIC AND STORAGE:

3 WORKING REGISTERS: B, b, c - half registers

RUNNING TIME:

2 seconds

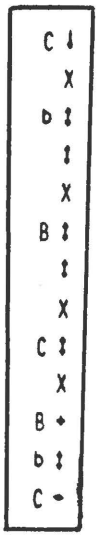
ARGUMENT(S):

X_1 in C
Y_1 in b
X_2 in M
Y_2 in B

FUNCTION(S) OR RESULT(S):

Re[W] in A
Im[W] in b

PROGRAM LISTING:



SAMPLE RUN:

```
                                    V

                  x1        5       S
                  Y1        0       S
                  x2       23       S
                  Y2        0       S

Re [W] =        115.000     A◊
Im [W] =          0.000     b◊

                                    0       S
                                    6       S
                                    0       S
                                   12       S

                                -72.000     A◊
                                  0.000     b◊

                                  2.3658    S
                                  1.3652    S
                                  6.1254    S
                                  3.8619    S

                        9.21920544000       A◊
                       17.49887910000       b◊
```

Title		
$\frac{Z_1}{Z_2}$ (complex)		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$W = \frac{Z_1}{Z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{z_1 x_2 + y_1 \; _2 + i\,(y_1 x_2 - y_2 x_1)}{x_2^{\,2} + y_2^{\,2}}$$

SPECIFICATION(S) ON ARGUMENT(S):

$Z_2 \neq 0$

LOGIC AND STORAGE:

4 WORKING REGISTERS: B, b, C, c - half registers

RUNNING TIME:

2 seconds

ARGUMENT(S):

X_1 in C
Y_1 in c
X_2 in B
Y_2 in b

FUNCTION(S) OR RESULT(S):

Re [W] in B
Im [W] in A

PROGRAM LISTING:

```
B ↓
C X
C ↕
b X
c ↕
  ↕
  X
C ↓
C ↑
R ◆
C ↕
B X
c -
B ↕
A X
b ↕
A X
b ◆
C ↕
C ↑
B ↕
C ↑
```

SAMPLE RUN:

$$W = \frac{x_1 + i\,y_1}{x_2 + i\,y_2}$$

V

$x_1 =$ 0 S
$y_1 =$ 7 S
$x_2 =$ 0 S
$y_2 =$ 28 S

Re [W] = 0.250 B 0
Im [W] = 0.000 A 0

4 S
0 S
2 S
0 S

2.000 B 0
0.000 A 0

7.56 S
2.03 S
3.65 S
1.95 S

1.842481751824 B 0
-0.428175182481 A 0

Title		
$\sqrt{Z}$ (Z complex)		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

$$W = \sqrt{Z} = \sqrt{x + iy} = \sqrt{R\cos\theta + iR\sin\theta}$$

$$= \sqrt{R}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]$$

$$= \sqrt{R}\left[\pm\sqrt{\frac{1+\cos\theta}{2}} \pm i\sqrt{\frac{1-\cos\theta}{2}}\right]$$

$$= \sqrt{R}\left[\pm\sqrt{\frac{1+\frac{x}{R}}{2}} \pm i\sqrt{\frac{1-\frac{x}{R}}{2}}\right]$$

$$= \pm\sqrt{\frac{R+x}{2}} \pm i\sqrt{\frac{R+x}{2}}$$

To obtain the principal root, an examination of the complex plane shows the choice of sign to be as follows:

$$\sqrt{\frac{R+X}{2}} + \text{sgn}(y)\sqrt{\frac{R-X}{2}}$$

where $\text{sgn}(y) = \begin{cases} +1 \text{ if } y \geq 0 \\ -1 \text{ if } y < 0 \end{cases}$

LOGIC AND STORAGE:

1 LABEL: fY
2 WORKING REGISTERS: b, B - half registers
1 CONSTANT REGISTER: (F = 2)

ERROR:

For X much larger than y there is a maximum possible error in the 7th decimal place of Im [W].

RUNNING TIME:

3 seconds

ARGUMENT(S):

x in B
y in b

FUNCTION(S) OR RESULT(S):

Re [W] in B
Im [W] in b

PROGRAM LISTING:

```
b ↓
A X
B ↕
A X
B ↓
B ↑
R •
A √
B -
F +
B ↕
B •
A √
B ↕
A √
b ↕
A S
F Y
R S
F Y
b ↓
A ↕
b ↕
f Y

2  F ◊
```

SAMPLE RUN:

$$W = \sqrt{x + iy}$$

```
                                    V

                  x =       25      S
                  y =        0      S

Re [W] =    5.0000000000    B◊
Im [W] =    0.0000000000    b◊

                             -25    S
                               0    S

                    0.0000000000    B◊
                    5.0000000000    b◊

                               0    S
                              25    S

                    3.5355339059    B◊
                    3.5355339059    b◊

                              25    S
                              25    S

                    5.4934205673    B◊
                    2.2754493028    b◊
```

Title
Z^u Recurrence (Z complex; n integral)

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

This routine performs the following recurrence:

$$\begin{cases} U_{n+1} = x\ U_n - y\ V_n \\ V_{n+1} = x\ V_n + y\ U_n \\ U_o = 1 \\ V_0 = 1 \end{cases}$$

$Z^u = U_n + i\ V_n$, where U_n and V_n are termed harmonic polynomials and the above is a recurrence scheme for integral power of a complex number

In matrix form the algorithm is simply:

$$\begin{aligned} U_n + 1 &= U_n - V_n \quad x \\ V_n + 1 &= V_n \quad U_n \quad y \end{aligned}$$

Various applications, from Fourier Analysis to extrapolation of equally spaced angles result. Some typical examples follow.

Sine and Cosine of multiple angles: sin(nx) and cosnx

The above recurrence is identical to the familiar "addition formula" used in developing the sine and cosine of multiple angles:

Given, $a = \sin(x)$
$b = \cos(x)$
$n_o = 0$

Then: $\sin[(n+1)] = a\cos(nx) + b\sin(nx)$
$\cos[(n+1)] = b\cos(nx) - a\sin(nx)$

SPECIFICATION(S) ON ARGUMENT(S):

sin (x) and cos (x) must be given as input

sin (nx) and cos (nx) are then computed for each pass, n, through the routine:

(n = 0, 1, 2, . . .)

LOGIC AND STORAGE:

2 WORKING REGISTERS: c, C

ERROR:

Provided that sin (x) and cos (x) are given correctly to 10 decimal places, then there is no error in sin (nx) and cos (nx) until n 50.

ARGUMENT(S):

sin (x) must be placed in Register b
cos (x) must be placed in Register B
Register C must initially contain 1
Register c must initially contain 0

FUNCTION(S) OR RESULT(S):

sin (nx) is left in Register c
cos (nx) is left in Register C

Sine and Cosine Extrapolation: sin (x + x) and cos (x + x)

Taking two terms* in the Maclaurin series for sine and cosine and using the appropriate additiona formulas, we obtain:

$$\sin(x_n + \Delta x) = \alpha \sin x_n + \beta \cos x_n$$
$$\cos(x_n + \Delta x) = \alpha \cos x_n - \beta \sin x_n,$$

where: $\alpha = 1 - \frac{(\Delta x)^2}{2!}$

$\beta = \Delta x - \frac{(\Delta x)^3}{3!}$

SPECIFICATION(S) ON ARGUMENT(S):

α, β, $\sin x_o$ and $\cos x_o$ must be given as input where Δx must be in radians. $\sin(x_n + \Delta x)$ and $\cos(x_n + \Delta x)$ are then computed for each pass, n, through the routine. The n^{th} argument may be written as $x_o + n\Delta x$.

LOGIC AND STORAGE:

2 WORKING REGISTERS: c, C
2 CONSTANT REGISTERS: (B) = α
(b) = β

ERROR:

Provided that x is small, the error remains small even for n very large.

$$\epsilon_n < \frac{v(v^n - 1)}{v-1}, \text{ where } v = \frac{(\Delta x)^4}{4!}$$

ARGUMENT(S):

α must be placed in Register B
β must be placed in Register b
$\sin x_o$ must be placed in Register c
$\cos x_o$ must be placed in Register C

FUNCTION(S) OR RESULT(S):

$\sin(x_n + \Delta x)$ is left in Register c
$\cos(x_n + \Delta x)$ is left in Register C

*if desired one may let $\alpha = \cos \Delta x$
$\beta = \sin \Delta x$

The routine as it stands would be useful for starting from a known Δx, $\sin x_o$, $\cos x_o$ and thus not having to compute $\cos \Delta x$, $\sin \Delta x$.

SUMMARY OF TYPICAL USES

Register	Z^n		Multiple Angles		Extropolation	
	Argument	Result	Argument	Result	Argument	Result
b	y		sin(x)		$\sin \Delta x$	
B	x		cos(x)		$\cos \Delta x$	
c	0	V	0	sin(nx)	$\sin X_o$	$\sin(X_o + n \Delta x)$
C	1	U	1	cos(nx)	$\cos X_o$	$\cos(X_o + n \Delta x)$

PROGRAM LISTING:

```
C↓
B X
b↕
 ↕
c↓
 X
 ↕
b↕
 -
C↕
b X
c↕
B X
c◆
c↕
```

SAMPLE RUN:

Z^n Recurrence

$Z = (x + i\,y)$

```
x =    1.245678   B↑
y =   -3.050505   b↑
               1  C↑
               0  c↑
```

```
                                  V
Z^1 {U1   -3.050505   c◊
    {V1    1.245678   C◊

Z^2 {U2   -7.599893   c◊
    {V2   -7.753867   C◊

Z^3 {U3   14.186189   c◊
    {V3  -32.842336   C◊

      117.857134   c◊
        2.364066   C◊
        .
        .
        .
      139.600442   c◊
      362.468644   C◊

     -931.815213   c◊
      877.371063   C◊

    -3837.166527   c◊
    -1749.585135   C◊
```

SAMPLE RUNS(Continued)

Sine and Cosine of multiple angles

```
sin(1 rad) =    0.8414709848   b↑
cos(1 rad) =    0.5403023059   B↑
                           0   c↑
                           1   C↑
```

```
                                V
sin(1 rad) =    0.8414709848   c◊
cos(1 rad) =    0.5403023059   C◊

sin(2 rad) =    0.9092974268   c◊
cos(2 rad) =   -0.4161468364   C◊

sin(3 rad) =    0.1411200081   c◊
cos(3 rad) =   -0.9899924966   C◊

sin(4 rad) =   -0.7568024952   c◊
cos(4 rad) =   -0.6536436209   C◊

               -0.9589242747   c◊
                0.2836621853   C◊

               -0.2794154983   c◊
                0.9601702866   C◊

                0.6569865986   c◊
                0.7539022545   C◊
```

Sine and Cosine Extrapolation: $\sin(x+n\Delta x)$ $\cos(x+n\Delta x)$

$x = 0^{\circ}.5 = 0.0087266462$ rad

```
sin(o.5) =   0.9999619229   B↑
cos(0.5) =   0.0087265355   b↑
sin(15°) =   0.2588190447   c↑
cos(15°) =   0.9659258264   C↑
```

```
                                V
sin(15°) =      0.2672383756   c◊
cos(15°) =      0.9636304531   C◊

sin(15°.5) =    0.2756373553   c◊
cos(15°.5) =    0.9612616957   C◊

sin(16°.0) =    0.2840153441   c◊
cos(16°.0) =    0.9588197345   C◊

                0.2923717041   c◊
                0.9563047554   C◊

                0.3007057988   c◊
                0.9537169500   C◊

                0.3090169936   c◊
                0.9510565154   C◊

                0.3173046556   c◊
                0.9483236542   C◊

                0.3255681536   c◊
                0.9455185744   C◊

                0.3338068583   c◊
                0.9426414898   C◊
```

Forward Recursion

Title		
e^x		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

Forward recursion:

$$e^x = \sum_{n=1}^{\infty} \frac{X^n}{n!}$$

SPECIFICATION(S) ON ARGUMENT(S):

$x \leq 34$

LOGIC AND STORAGE:

1 LABEL: aV
3 WORKING REGISTERS: b, B, c - half registers
1 CONSTANT REGISTER: (F)= 1

ERROR:

For $|x| < 1$, there is an error in the last decimal place computed

RUNNING TIME:

14 seconds for $|x| < 1$

ARGUMENT(S):

x in b

FUNCTION(S) OR RESULT(S):

x in b
e^x in B

PROGRAM LISTING:

```
c *
F •
B ↑
aV
  ↓
c ↕
F •
c ↕
c ↕
bX
B ↕
B •
B ↕
A ↓
AS
AV
```

1 F◊

SAMPLE RUN:

```
                                   V
x                          0.1     S
e^x   1.1051709180756 1         B◊

x                          0.02    S
e^x   1.0202013400267 3         B◊

                           0.3     S
      1.3498588075759 8         B◊

                          -0.123   S
      0.8842636625608 2         B◊

                          -1       S
      0.3678794411714 4         B◊

                           1       S
      2.7182818284589 8         B◊
```

Title		
sin x		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

Forward Recursion based on the series

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

SPECIFICATION(S) ON ARGUMENT(S):

x in radians

LOGIC AND STORAGE:

1 LABEL: aV
1 GENERATED CONSTANT: 2
3 WORKING REGISTERS: b, B, c - half registers

ERROR:

2 in the last decimal place for $|x| < \pi/2$

RUNNING TIME:

10 seconds for $|x| < \pi/2$

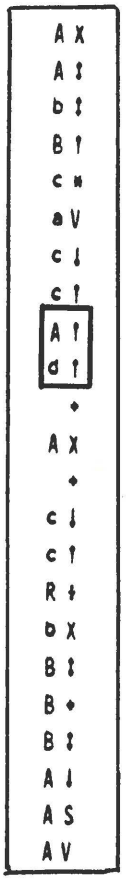
ARGUMENT(S):

x in A

FUNCTION(S) OR RESULT(S):

sin x is left in B

PROGRAM LISTING:



SAMPLE RUN:

```
                                                    V

x          0.785398163397 45                        S
sin x      0.707106781186 57                        B 0

x          1.047197551196 60                        S
sin x      0.866025403784 44                        B 0

           1.570796326794 90                        S
           1.000000000000 00                        B 0

                                                0.1 S
           0.099833416646 83                        B 0

                                                0.2 S
           0.198669330795 06                        B 0

                                                0.3 S
           0.295520206661 35                        B 0

                                               -0.1 S
          -0.099833416646 83                        B 0

                                                  1 S
           0.841470984807 91                        B 0
```

Title		
cos x		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

Forward Recursion:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

LOGIC AND STORAGE:

1 LABEL: aV
2 GENERATED CONSTANTS: 1, 2
3 WORKING REGISTERS: b, B, c - half registers

ERROR:

for $|x| < \frac{\pi}{2}$, error $<$ 2 in last place calculated.

RUNNING TIME:

$|x| < \frac{\pi}{2}$ 10 seconds

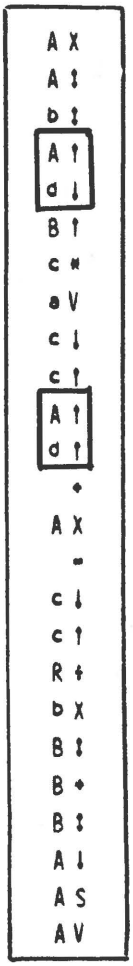
ARGUMENT(S):

x in A

FUNCTION(S) OR RESULT(S):

cos x in B

PROGRAM LISTING:



SAMPLE RUN:

```
                                          V
x          0.785398163397 45              S
cos x      0.707106781186 55              B 0

x          1.047197551196 60              S
cos x      0.500000000000 00              B 0

           1.570796326794 90              S
         - 0.000000000000 01              B 0

                           0.1            S
           0.995004165278 02              B 0

                           0.2            S
           0.980066577841 25              B 0

                           0.3            S
           0.955336489125 61              B 0

                         - 0.1            S
           0.995004165278 02              B 0

                             1            S
           0.540302305868 13              B 0
```

Title		
sinh x		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

Forward recursion

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

SPECIFICATION(S) ON ARGUMENT(S):

$|x| \leq 34$

LOGIC AND STORAGE:

1 LABEL: aV
1 GENERATED CONSTANT: 2
3 WORKING REGISTERS: b, B, c - half registers

ERROR:

2 in the last significant digit computed.

RUNNING TIME:

for $|x| < 2$, 12 seconds

ARGUMENT(S):

x in A

FUNCTION(S) OR RESULT(S):

sinh x in B

PROGRAM LISTING:

```
A X
D ↕
B ↑
c ∗
a V
c ↓
c ↑
A ↑
d ↑
  ◆
A X
  ◆
c ↓
c ↑
R ↓
b X
B ↕
B ◆
B ↕
A ↓
A S
A V
```

SAMPLE RUN:

```
                                    V
x                                   1    S
sinh x      1.175201193643 77       B 0

x                                 0.1    S
sinh x      0.100166750019 83       B 0

                                   10    S
      11013.232874702400 00         B 0

                                   -1    S
     -1.175201193643 77             B 0

                                0.00001  S
      0.000010000000 00             B 0
```

Title
cosh x

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

Forward recursion

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

SPECIFICATION(S) ON ARGUMENT(S):

$|x| \leq 34$

LOGIC AND STORAGE:

1 LABEL: aV
2 GENERATED CONSTANT: b, B, C - half registers

ERROR:

2 in the last significant digit computed

RUNNING TIME:

11 seconds for $|x| < 2$

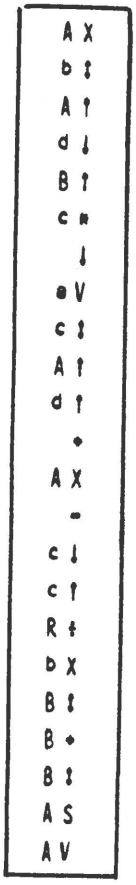
ARGUMENT(S):

x in A

FUNCTION(S) OR RESULT(S):

cosh x in B

PROGRAM LISTING:



SAMPLE RUN:

```
                                                V
x                                          1    S
cosh x      1.543080634815 21              B 0

x                                          2    S
cosh x      3.762195691083 44              B 0

                                          10    S
      11013.232920102400 00                B 0

                                      0.0001    S
      1.000000005000 00                    B 0

                                          -1    S
      1.543080634815 21                    B 0

                                        -0.5    S
      1.127625965206 35                    B 0
```

Title
$(1 + i)^{k/m}$

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

A forward recursion based on the binomial expansion

$$(1 + i)^{k/m} = 1 + \sum_{n=1}^{\infty} \left[\frac{(-k)}{m} \frac{(m-k)}{2m} \frac{(2m-k)}{3m} \cdots \frac{(n-1)m}{nm} \right] (-i)^n$$

SPECIFICATION(S) ON ARGUMENT(S):

$|x| < 1$

LOGIC AND STORAGE:

2 LABELS:	aY, aZ
1 GENERATED CONSTANT:	1
6 WORKING REGISTERS:	B, C, E - half registers
	d, D, e - quarter registers

ERROR:

There is an error in the last decimal place calculated.

RUNNING TIME:

10 seconds for $|x| < 0.1\,k < m$

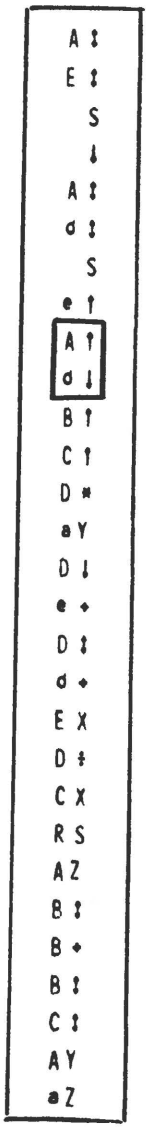
ARGUMENT(S):

-k in d
m in e
-x in E

FUNCTION(S) OR RESULT(S):

$(1+x)^{k/m}$ is left in B

PROGRAM LISTING:



SAMPLE RUN:

```
                                                    V

                  i                    0.01         S
                  k                      14         S
                  m                      30         S
(1 + i)         k/m    1.00465428542714   B0

                  i                     0.1         S
                  k                       5         S
                  m                      12         S
(1 + i)         k/m    1.04051166205899   B0

                                       0.12         S
                                         11         S
                                         12         S
                       1.10947244569617   B0

                                      0.125         S
                                         49         S
                                         90         S
                       1.06622707543338   B0
```

Backward Recursion

Title
e^x

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

Backward Recursion based on series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

SPECIFICATION(S) ON ARGUMENT(S):

The number of terms calculated, 17, is sufficient for the range $|x| < 1$. For $|x| > 1$ more terms are necessary.

LOGIC AND STORAGE:

1 LABEL: aV
2 GENERATED CONSTANTS: 17, 1
2 WORKING REGISTERS: b, B - half registers

ERROR:

Error < 2 in the last decimal place computed

RUNNING TIME:

15 seconds

ARGUMENT(S):

x in B

FUNCTION(S) OR RESULT(S):

e^x is left in B
x is left in b

PROGRAM LISTING:

```
 b ↑
[A ↑]
[D ÷]
[r ↓]
   ↓
 B *
 a V
 B ↕
 B ÷
 b X
[A ↑]
[a ↓]
   +
 B ↕
   -
 A S
 A V
```

SAMPLE RUN:

```
                              V

x                        1    S
e^x   2.718281828459045      B ◊

x                      0.1    S
e^x   1.105170918075648      B ◊

                       0.2    S
      1.221402758160169      B ◊

                       0.3    S
      1.349858807576003      B ◊

                      -0.5    S
      0.606530659712633      B ◊

                        -1    S
      0.367879441171442      B ◊
```

Title
sin x

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

A backward recursion based on the continued fraction expansion.

$$\sin x = \cfrac{x}{1+\cfrac{x^2}{(2\cdot3 - x^2)+\cfrac{2\cdot3\,x^2}{(4\cdot5 - x^2)+\cfrac{4\cdot5\,x^2}{(6\cdot7 - x^2)+}}}} \quad \cdots$$

LOGIC AND STORAGE:

1 LABEL:	aV
2 GENERATED CONSTANTS:	20, 1
3 WORKING REGISTERS:	b, B, c, - half registers
1 CONSTANT REGISTER:	(F) = 2

ARGUMENT(S):

x in B

FUNCTION(S) OR RESULT(S):

sin x in A
x in b

PROGRAM LISTING:

```
A X
b ↑
B ↕
[ A ↑
  D S
  r ↑ ]
c ↑
R ↑
a W
B ↓
R +
c ↕
F -
A X
  ◊
R S
A V
c ↓
c ↑
R X
  ◊
B -
A W
a V
[ A ↑
  d ↓ ]
c ↓
  ◊
b ↓
R +

2   F ◊
```

SAMPLE RUN:

```
                                        V

x =         0.01                        S
sin x =     0.00999983333416            A ◊

            0.02                        S
            0.01999866669333            A ◊

            0.03                        S
            0.02999550020249            A ◊

            0.04                        S
            0.03998933418663            A ◊

            0.05                        S
            0.04997916927067            A ◊

            0.1                         S
            0.09983341664682            A ◊

            0.5                         S
            0.47942553860420            A ◊

            1                           S
            0.84147098480789            A ◊

            1.04719755119659            S
            0.86602540378443            A ◊

            1.57079632679489            S
            1.00000000000000            A ◊

            -1.5                        S
           -0.99749498660405            A ◊
```

Title		
tan x		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

A continued fraction expansion with a specified number of terms.

$$\tan(x) = \frac{x}{1 -} \quad \frac{x^2}{3 -} \quad \frac{x^2}{5 -} \quad \frac{x^7}{7 -} \quad \cdots$$

SPECIFICATION(S) ON ARGUMENT(S):

X in Radians, $X \neq \frac{\pi}{2} + n\pi$

LOGIC AND STORAGE:

1 LABEL: aV
2 GENERATED CONSTANTS (19) (Represents 2n-1 where n is number terms calculated)
3 WORKING REGISTERS: b, B, C - half registers

ERROR:

For $|X| < \frac{\pi}{2}$ there is an error in the last decimal place calculated.

RUNNING TIME:

17 seconds

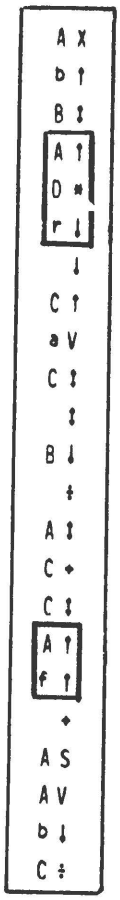
ARGUMENT(S):

x in A

FUNCTION(S) OR RESULT(S):

tan x is left in A

PROGRAM LISTING:



SAMPLE RUN:

```
                                        V
x                              0.1      S
tan (x)      0.10033467208544           A◊

x                              0.2      S
tan (x)      0.20271003550867           A◊

                               0.3      S
             0.30933624960962           A◊

                               0.4      S
             0.42279321873815           A◊

                               0.5      S
             0.54630248984378           A◊

                              -0.25     S
            -0.25534192122103           A◊

                               1.57     S
          1255.76559153133000           A◊

                               0.785    S
             0.99920399010504           A◊
```

Title		
Arctangent: $\tan^{-1}(x)$		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

A continued fraction expansion with 19 terms

$$\tan^{-1}(x) = \cfrac{x}{1+\cfrac{x^2}{3+\cfrac{(2x)^2}{5+\cfrac{(3x)^2}{7+\ldots}}}}$$

LOGIC AND STORAGE:

1 LABEL: AV
2 GENERATED CONSTANTS: 19, 1
3 WORKING REGISTERS: b, B, c - half registers

ERROR:

There is an error in the last decimal place computed for x

RUNNING TIME:

17 seconds

ARGUMENT(S):

x in b

FUNCTION(S) OR RESULT(S):

$\tan^{-1}(x)$ is left in A
X is left in b

PROGRAM LISTING:

```
A ↑
D *
r ↓
c ↑
A *
a V
B ↕
c ↓
A ↑
d ↓
  -
c ↕
c ◆
B ◆
B ↕
c ↓
b X
A X
B ÷
A S
A V
b ↓
  ÷
```

SAMPLE RUN:

x =	0.1	S
Arctan (x) =	0.09966865249116	A ◊
	0.2	S
	0.19739555984988	A ◊
	0.3	S
	0.29145679447786	A ◊
	0.4	S
	0.38050637711236	A ◊
	0.5	S
	0.46364760900080	A ◊
	0.6	S
	0.54041950027058	A ◊
	0.7	S
	0.61072596438920	A ◊
	0.8	S
	0.67474094222355	A ◊
	0.9	S
	0.73281510178651	A ◊
	1	S
	0.78539816339745	A ◊

Other Algorithms

Title		
Exponential X^n		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

The following recursion is used:

$$Y_{i+1} = [X_i]^{e_i} Y_i \quad (e_i = 0, 1)$$

$$X_{i+1} = X_i^2$$

$$Y_o = 1$$

$$X_o = X$$

SPECIFICATION(S) ON ARGUMENT(S):

n is a non-negative integer.

LOGIC AND STORAGE:

2 LABELS:	aV, aW, bV
2 GENERATED CONSTANTS:	1, 2
2 WORKING REGISTERS:	C, D - half registers

ERROR:

If the full result does not exceed the D.W.S., there is no error. Error depends on number of multiplications where result exceeds D.W.S.

RUNNING TIME:

Depends on n. For $n < 300$, running time is less than 6 seconds.

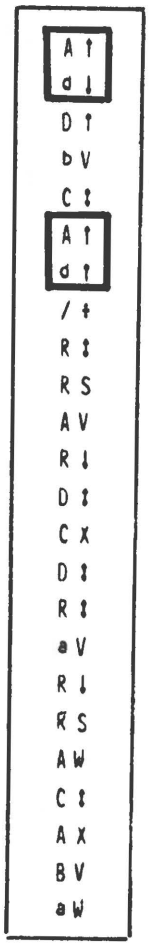
ARGUMENT(S):

X in A
n in C

FUNCTION(S) OR RESULT(S):

X^n is left in D

PROGRAM LISTING:



SAMPLE RUN:

```
                                        V
x                             1.01      S
n                              300      S
x^n          19.788466261920 20        D◊

x                                2      S
n                               18      S
x^n   262144.000000000000000           D◊

x                                3      S
n                                2      S
x^n           9.000000000000000        D◊
```

Real-Valued Functions

Title		
Gauss Hypergeometric Series ${}_2F_1(\alpha, \beta; \gamma; x)$		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

A forward recursion based on the series:

$$ {}_2F_1(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\ \beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)} \cdot \frac{x^n}{n!} $$

SPECIFICATION(S) ON ARGUMENT(S):

$|x| < 1$

LOGIC AND STORAGE:

2 LABELS:	aV, aW
6 WORKING REGISTERS:	b, B, c, C, d, D - half registers
1 CONSTANT REGISTER:	(F) = 1

ERROR:

In general there is an error in the last place calculated.

RUNNING TIME:

Depends on size of x

ARGUMENT(S):

α in b γ in C
β in B x in D

FUNCTION(S) OR RESULT(S):

${}_2F_1(\alpha, \beta; \gamma; x)$ is left in d.
x is left in D.

PROGRAM LISTING:

```
A *
F ↑
d ↑
C ↑
aW
b X
B X
c ↕
C ↕
D X
R S
A V
d ↕
d ↑
d ↕
b ↕
F ↑
B ↕
  ↑
B ↕
b ↕
c ↕
  ↑
c ↕
C ↕
  ↑
C ↕
A W
aV
```

SAMPLE RUN:

```
                                                V
                          α                 1   S
                          β                 1   S
                          γ                 2   S
                          x              -0.5   S
2F1(α, β; γ; x)   0.810930216216333   d ◊

                          α               0.5   S
                          β                 1   S
                          γ               1.5   S
                          x              0.01   S
2F1(α, β; γ; x)   1.003353477310732   d ◊

                                          0.5   S
                                            1   S
                                          1.5   S
                                         0.25   S
                  1.098612288667990   d ◊

                                          0.5   S
                                          0.5   S
                                          1.5   S
                                         0.25   S
                  1.047197551196498   d ◊
```

Title
Confluent Hypergeometric Function ${}_1F_1(\alpha;\beta;x)$

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

A forward recursion based on the series:

$$ {}_1F_1(\alpha;\beta;x) = \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{\beta(\beta+1)\ldots(\beta+n-1)} \frac{x^n}{n!} $$

LOGIC AND STORAGE:

2 LABELS: aV, aW
5 WORKING REGISTERS: b, B, c, C, d - half registers
1 CONSTANT REGISTER: F = 1

ERROR:

The error varies with the number of terms of the series computed.

RUNNING TIME:

Varies with the number of terms computed.

ARGUMENT(S):

α in b X in d
β in B

FUNCTION(S) OR RESULT(S):

${}_1F_1(\alpha;\beta;x)$ is left in C

PROGRAM LISTING:

```
A *
F ◆
C ↑
c ↑
a W
b X
B +
c +
d X
R S
A V
C ↕
C ◆
C ↕
b ↕
F ◆
b ↕
B ↕
  ◆
B ↕
c ↕
  ◆
c ↕
A W
a V

1   F ◊
```

SAMPLE RUN:

			V
	α	1	S
	β	2	S
	x	1.4	S
${}_1F_1\,(\alpha;\beta;x)$		2.18228569060319	C ◊
	α	0.5	S
	β	1.5	S
	x	-0.049	S
${}_1F_1\,(\alpha;\beta;x)$		0.98390399197637	C ◊
		1.1	S
		2.2	S
		1.4	S
		2.17160419828880	C ◊
		1	S
		1	S
		-0.7	S
		0.49658530379140	C ◊
		1	S
		1	S
		1	S
		2.71828182845898	C ◊

Title		
Bessel Functions of Integer order $J_n(x)$		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

A backward recursion based on the Taylor series:

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \left(-\frac{x}{4}\right)^k \frac{1}{k!\,(n+k)!}$$

20 terms are calculated. For faster but less accurate routines, the first number generation should be changed to a lower number.

LOGIC AND STORAGE:

4 LABELS:	bV, rV, cW, bZ
2 GENERATED CONSTANTS:	20, 2
4 WORKING REGISTERS:	b, B, c, C - half registers
1 CONSTANT REGISTER:	f = 1

ERROR:

There is an error in the last decimal place calculated.

RUNNING TIME:

About 40 seconds.

ARGUMENT(S):

n in A, X in C/

FUNCTION(S) OR RESULT(S):

$J_n(x)$ is left in C

PROGRAM LISTING:

```
A ↑
D S
r ↑
  ◇
c ↕
C ↑
A ↑
d ↑
  +
A X
A ↕
b ↑
B ↕
b V
B X
c +
C +
f ◇
c ↕
  -
c ↕
C ↕
  -
R S
R V
C ↕
B V
r V
c ↓
R S
B Z
c W
C ↕
b X
C +
C ↕
f -
A S
C W
b Z
```

SAMPLE RUN:

```
                                V
n                       1       S
x                     2.9       S
J_1(x)  0.3754274818131 0       C ◇

                                V
n                       0       S
x                     2.1       S
J_0(x)  0.1666069803319 9       C ◇

                        9       S
                        1       S
        0.0000000005249 25      C ◇

                        5       S
                        5       S
        0.2611405461201 5       C ◇

                        0       S
                      4.5       S
       -0.3205425089851 3       C ◇
```

Title
Kelvin Function, Zero Order ber (x)

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

A forward recursion based on the series:

$$\text{ber}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{4n}}{[(2n)!]^2}$$

LOGIC AND STORAGE:

1 LABEL: aV
1 GENERATED CONSTANT: 1
3 WORKING REGISTERS: b, B, c - half registers

ERROR:

There is an error in the last decimal place calculated.

RUNNING TIME:

6 seconds at x = 1.

ARGUMENT(S):

x in A

FUNCTION(S) OR RESULT(S):

ber (x) is left in B

PROGRAM LISTING:

```
F ↑
A X
A X
A ↕
b ↕
A ↑
d ↓
B ↑
c ×
aV
  ↓
c ↕
F -
A X
  •
c ↓
c ↑
R ↑
  +
b X
B ↕
B •
B ↕
A ↓
A S
A V
```

2 F 0

SAMPLE RUN:

```
                x =                0.1    S
ber (x) = 0.999999843750006        B 0

                                   0.2    S
          0.999997500001736        B 0

                                   0.3    S
          0.999987343794494        B 0

                                   0.4    S
          0.999960000444444        B 0

                                   0.5    S
          0.999023463990 84        B 0

                                     1    S
          0.984381781213 07        B 0

                                     2    S
          0.751734182713 81        B 0

                                     3    S
         -0.221380249598 71        B 0

                                     4    S
         -2.563416557258 72        B 0

                                     5    S
         -6.230082478666 40        B 0
```

Title
Kelvin Function, Zero Order bei (x)

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

A forward recursion based on the series:

$$\text{bei}\,(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{x}{2})^{4n+2}}{[(2n+1)!]^2}$$

LOGIC AND STORAGE:

1 LABEL: aV
2 GENERATED CONSTANTS: 4, 2
3 WORKING REGISTERS: b, B, c

ERROR:

There is an error in the last decimal place computed.

RUNNING TIME:

6 seconds at x = 1

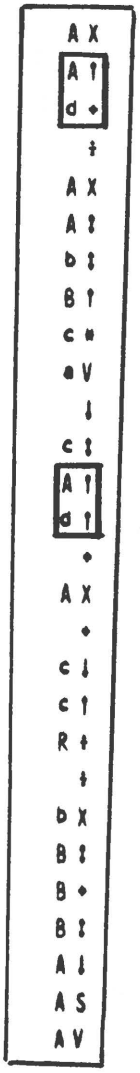
ARGUMENT(S):

x in A

FUNCTION(S) OR RESULT(S):

bei x is left in B

PROGRAM LISTING:



SAMPLE RUN:

```
                                                    V
          x =                              0.1      S
bei (x)= 0.0024999999656598                B0

                                           0.2      S
          0.0099999972222223               B0

                                           0.3      S
          0.022499683594l5                 B0

                                           0.4      S
          0.039998222229 34                B0

                                           0.5      S
          0.062493218382 20                B0

                                           1        S
          0.249566040036 67                B0

                                           2        S
          0.972291627306 67                B0

                                           3        S
          1.937586785266 08                B0

                                           4        S
          2.292690322699 20                B0

                                           5        S
          0.116034381549 62                B0
```

Title		
Definite integral of the Bessel Function $I_o(x)$		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

A forward recursion based on the series:

$$f(x) = \int_o^x I_o(t)\,dt = \sum_{n=0}^{\infty} \frac{1}{(n!)^2\,(2n+1)} \left(\frac{x}{2}\right)^{2n}$$

LOGIC AND STORAGE:

1 LABEL: aV
2 GENERATED CONSTANTS: 1, 2
3 WORKING REGISTERS: b, B, c

ERROR:

There is an error in the last decimal place computed.

RUNNING TIME:

9 seconds at x = 1

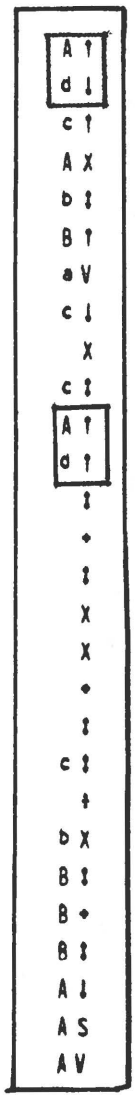
ARGUMENT(S):

x in A

FUNCTION(S) OR RESULT(S):

f(x) is left in B

PROGRAM LISTING:



SAMPLE RUN:

```
x =                                0.1    S
f(x) =    0.100083364589 52       B 0

                                   0.2    S
          0.200667667460 68       B 0

                                   0.3    S
          0.302257607325 10       B 0

                                   0.4    S
          0.405365435118 39       B 0

                                   0.5    S
          0.510514808797 37       B 0

                                     1    S
          1.086521097023 55       B 0

                                     2    S
          2.775001905428 04       B 0

                                     3    S
          6.160961491501 80       B 0

                                     4    S
         13.775208868038 20       B 0

                                     5    S
         31.848667776167 30       B 0

                                    -1    S
         -1.086521097023 55       B 0
```

Title
Dilogarithm

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

A forward recursion based on the series:

$$f(x) = \sum_{n=1}^{\infty} \frac{(1-x)^n}{n^2} = -\int_1^x \frac{\ln(t)}{t-1}\,dt$$

SPECIFICATION(S) ON ARGUMENT(S):

0 x 2

LOGIC AND STORAGE:

1 LABEL: aV
3 WORKING REGISTERS: b, B, c - half registers
1 CONSTANT REGISTER:

ERROR:

There is an error in the next to last decimal place calculated.

RUNNING TIME:

45 seconds at x = 1.5

ARGUMENT(S):

x in A

FUNCTION(S) OR RESULT(S):

f(x) is left in B

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PROGRAM LISTING:

```
F −
A ↕
c ↑
  ↕
b ↑
B ↑
a V
c ↓
  ×
c ×
c ↕
F ÷
A ×
  ↕
c ↕
  +
b ×
B ↕
B ÷
B ↕
A ↓
A S
A V
```

```
1   F ◊
```

SAMPLE RUN:

```
                                          V

   x =                    0.5             S
f(x) =     0.58224052646467               B ◊

                          0.6             S
           0.44928297447103               B ◊

                          0.7             S
           0.32612951007533               B ◊

                          0.8             S
           0.21100377543959               B ◊

                          0.9             S
           0.10261779109932               B ◊

                            1             S
          -0.00000000000000               B ◊

                          1.1             S
          -0.09760523522934               B ◊

                          1.2             S
          -0.19080013777753               B ◊

                          1.3             S
          -0.28007433375959               B ◊

                          1.4             S
          -0.36583257751245               B ◊

                          1.5             S
          -0.44841420692363               B ◊
```

Title
Error Function: Erf (x)

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

A forward recursion based on the series:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(2n+1)}$$

LOGIC AND STORAGE:

1 LABEL:	aV
1 GENERATED CONSTANT:	-1
3 WORKING REGISTERS:	b, B, c - half registers
1 CONSTANT REGISTER:	F = $\sqrt{\pi}$

ERROR:

There is an error in the last decimal place computed.

RUNNING TIME:

23 seconds at x = 1

ARGUMENT(S):

x in A

FUNCTION(S) OR RESULT(S):

erf (x) is left in A

PROGRAM LISTING:

```
A ↑
d ↓
c ↑
A •
  ↕
  X
A ↕
B ↑
b ↕
a V
  ↓
c X
b X
c ↕
A ↑
d ↑
  •
A X
  -
c ↓
c ↑
R +
B ↕
B •
B ↕
A ↓
A S
A V
B ↓
F +
```

1.77245385090551 F◊

SAMPLE RUN:

```
                                  V
x =                          0.1  S
erf (x)= 0.112462916018 28   A◊

                             0.2  S
         0.222702589210 47   A◊

                             0.3  S
         0.328626759459 12   A◊

                             0.4  S
         0.428392355046 67   A◊

                             0.5  S
         0.520499877813 04   A◊

                            0.01  S
         0.011283415555 84   A◊

                               0  S
         0.000000000000 00   A◊

                              -1  S
        -0.842700792949 71   A◊
```

Title
Fresnel Integral: S(x)

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

A forward recursion based on the expansion:

$$S(x) = \int_0^x \sin\left(\frac{\pi}{2} t^2\right) dt = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{2}\right)^{2n}}{(2n+1)!\,(4n+3)} X^{4n+3}$$

LOGIC AND STORAGE:

1 LABEL:	aV
2 GENERATED CONSTANTS:	3, 4
3 WORKING REGISTERS:	b, B, c - half registers
1 CONSTANT REGISTER:	(F) = $\frac{\pi}{2}$

ERROR:

There is an error in the last place calculated.

RUNNING TIME:

15 seconds at x = 1

ARGUMENT(S):

x in A

FUNCTION(S) OR RESULT(S):

S(x) is left in B

PROGRAM LISTING:

```
A X
B ↑
F X
A ◆
b ↑
A X
A ↕
b ↕
B X
A ↑
d ↕
c ↑
  +
B ↕
B ◆
a V
c ↓
  X
c ↕
A ↑
d ◆
  ↕
  ◆
  ↕
  X
  X
  ◆
  ◆
  ◆
  ↕
c ↕
  +
b X
B ↕
B ◆
B ↕
A ↓
A S
A V
```

SAMPLE RUN:

```
                                        V

   x                          0.1       S
S(x)     0.00052358954761              B◊

                              0.2       S
         0.00418760916165              B◊

                              0.3       S
         0.01411699800658              B◊

                              0.4       S
         0.03335943266061              B◊

                              0.5       S
         0.06473243286000              B◊

                              0.6       S
         0.11054020735941              B◊

                              0.7       S
         0.17213645786345              B◊

                              0.8       S
         0.24934139305391              B◊

                              0.9       S
         0.33977634439321              B◊

                                1       S
         0.43825914739030              B◊
```

```
1.57079632679489   F◊
```

Title
Fresnel Integral: C(x)

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

A forward recursion based on the series:

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2} t^2\right) dt = \sum_{n=0}^{\infty} \frac{x(-\pi^2 X^4)^n}{(2n)!(4)(4n+1)}$$

LOGIC AND STORAGE:

1 LABEL:	aV
2 GENERATED CONSTANTS:	1, 4
3 WORKING REGISTERS:	b, B, c - half registers
1 CONSTANT REGISTER:	$F = \pi^2$

ERROR:

There is an error in the last significant digit computed.

RUNNING TIME:

13 seconds at x = 1

ARGUMENT(S):

x in B

FUNCTION(S) OR RESULT(S):

C(x) is left in A

PROGRAM LISTING:

```
B↓
AX
AX
FX
b↕
A↑
d↓
c↑
B◆
eV
 ↓
cX
c↕
A↑
d◆
 ↕
 ◆
 ↕
 X
 X
 ◆
 ◆
 ◆
 ↕
c↕
 ◆
bX
B↕
B◆
B↕
A↓
AS
AV
```

SAMPLE RUN:

```
                                   V
  x                          0.1   S
C(x)  0.099997532627 08      B0

                           0.2     S
      0.19992105759446     B0

                           0.3     S
      0.29940097605205     B0

                           0.4     S
      0.39748075917236     B0

                           0.5     S
      0.49234422587145     B0

                           0.6     S
      0.58109544699164     B0

                           0.7     S
      0.65965235190452     B0

                           0.8     S
      0.72264417189632     B0

                           0.9     S
      0.76482302127321     B0

                           1       S
      0.77989340037682     B0
```

-9.86960440108933 F0

Title
Sine Integral $Si(x)$

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

A forward recursion based on the series:

$$Si(x) = \sum_{0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} = \int_{0}^{x} \frac{\sin(t)}{t} \, dt$$

LOGIC AND STORAGE:

1 LABEL: aV
2 GENERATED CONSTANTS: 1, 2
3 WORKING REGISTERS: b, B, c - half registers

ERROR:

For $|x| \leq 1$
There is an error in the last decimal place

RUNNING TIME:

11 seconds for $|x| \leq 1$

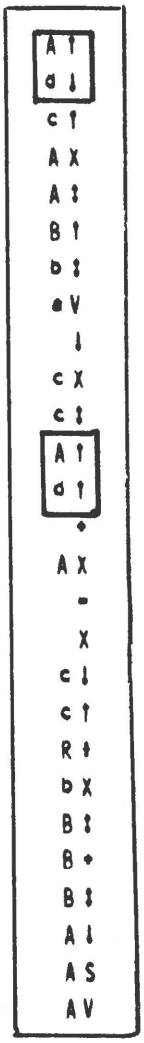
ARGUMENT(S):

x in A

FUNCTION(S) OR RESULT(S):

Si(x) is left in B

PROGRAM LISTING:



SAMPLE RUN:

```
                                                    V

                                        x = 0.01    S
Si(x) =     0.00999994444461            B0

                                        0.05        S
            0.04999305607637            B0

                                        0.1         S
            0.09994446110828            B0

                                        0.2         S
            0.19955608852623            B0

                                        0.3         S
            0.29850404380705            B0

                                        0.4         S
            0.39646146475138            B0

                                        0.5         S
            0.49310741804307            B0

                                        1           S
            0.94608307036718            B0
```

Title
Cosine Integral Cin(x)

Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

A forward recursion based on the series:

$$\mathrm{Cin}(x) = -\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!\,2n} = \int_0^x (1 - \cos t)\ \frac{dt}{t}$$

LOGIC AND STORAGE:

1 LABEL: aV
2 GENERATED CONSTANTS: (-2), (1)
3 WORKING REGISTERS: b, B, c - half registers

ERROR:

For $|x| < 1$ there is an error in the last decimal place.

RUNNING TIME:

11 seconds at x = 1

ARGUMENT(S):

x in A

FUNCTION(S) OR RESULT(S):

Cin(x) is left in B

PROGRAM LISTING:

```
 A X
[A ↑]
[f ↑]
   ÷
 b ↕
 b ↓
   ÷
 B ↕
[A ↑]
[d ↓]
 c ↑
 B ◆
 a V
   ↓
 c X
 c ↕
 / ÷
   ◆
   ↕
   ◆
   X
   X
 c ↓
 c ↑
 R ÷
 b X
 B ↕
 B ◆
 B ↕
 A ↓
 A S
 A V
```

SAMPLE RUN:

```
                                    V
          x =          0.01         S
Cin (x)=  0.0000249998958 4         B ◊

                       0.02         S
          0.0000999983333 5         B ◊

                        0.1         S
          0.0024989585647 9         B ◊

                        0.2         S
          0.0099833481402 2         B ◊

                        0.3         S
          0.0224157935467 6         B ◊

                        0.4         S
          0.0397342794526 2         B ◊

                        0.5         S
          0.0618525631482 0         B ◊

                          1         S
          0.2398117420005 6         B ◊

                        1.5         S
          0.5123244558143 2         B ◊

                          3         S
          1.5561981675617 0         B ◊

                          4         S
          2.1044917239082 4         B ◊
```

Title		
Hermite Polynomials $H_n(x)$		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

The recursion:

$$H_{k+1}(x) = 2\,(x\,H_k(x) - k\,H_{k-1}(x))$$

is used with

$$H_{-1} = 0$$
$$H_0 = 1$$

SPECIFICATION(S) ON ARGUMENT(S):

$n > 0$

LOGIC AND STORAGE:

1 LABEL: aV
5 WORKING REGISTERS: b, B, c, C, d - half registers
1 CONSTANT REGISTER: (F)= 1

ERROR:

For x an integer the polynomials are integers, hence exact. Non-integer x will result in a small error for large n.

RUNNING TIME:

varies with n and x

ARGUMENT(S):

n in c
x in d

FUNCTION(S) OR RESULT(S):

$H_n(x)$ is left in b
n and x are undisturbed

PROGRAM LISTING:

```
F ↓
b ↕
c ■
a V
c ↓
B X
B ↕
b ↓
d X
B -
A +
b ↕
B ↕
c ↓
F +
c ↕
C ↓
c -
A S
A V
```

1 F ◊

SAMPLE RUN:

```
                              V
n                 8           S
x                 3           S
H_n(x)        36240           b ◊

                  3           S
                  5           S
                940           b ◊

                 11           S
                 10           S
153373602947200               b ◊
```

Title		
Hermite Polynomial $He_n(x)$		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

The recursion:

$$He_{k+1}(x) = XHe_k(x) - kHe_{k-1}(x)$$

SPECIFICATION(S) ON ARGUMENT(S):

$n > 0$

LOGIC AND STORAGE:

1 LABEL:	aV
5 WORKING REGISTERS:	b, B, c, C, d - half registers
1 CONSTANT REGISTER:	(F) = 1

ERROR:

For x an integer the polynomials are integers, hence exact. Non-integer x will result in a small error for large n.

RUNNING TIME:

varies with n and x

ARGUMENT(S):

n in C
x in d

FUNCTION(S) OR RESULT(S):

$He_n(x)$ is left in b
n and x are undistrubed.

PROGRAM LISTING:

```
F ↓
b ↕
c *
a V
c ↓
B X
B ↕
b ↓
d X
B -
b ↕
B ↕
c ↓
F +
c ↕
C ↓
c -
A S
A V
1  F 0
```

SAMPLE RUN:

```
                                V
n                       5       S
x                       3       S
He_n(x)                 18      b 0

                        10      S
                        5       S
                    179680      b 0

                        10      S
                        14      S
              227467669891      b 0

                        14      S
                        10      S
          34831450009365        b 0

                        15      S
                        10      S
      2909026759547 50          b 0

                        10      S
                        1       S
                    1216        b 0
```

Title		
Generalized Laguerre Polynomial $L_n^{(\alpha)}(x)$		
Number of Sides	Lower Decimal Wheel	Upper Decimal Wheel

METHOD:

The recursion:

$$(k+1)\,L_{k+1}^{(\alpha)}(x) = (2k+\alpha+1)\,L_k^{(\alpha)}(x) - (k+\alpha)\,L_{k-1}^{(\alpha)}(x)$$

is used with

$$L_0^{(\alpha)}(x) = 1$$

$$L_1^{(\alpha)}(x) = \alpha + 1 - x$$

SPECIFICATION(S) ON ARGUMENT(S):

$n \geq 2$

LOGIC AND STORAGE:

1 LABEL:	aV
6 WORKING REGISTERS:	b, B, c, C, d, D - half registers
1 CONSTANT REGISTER:	F = 1

ERROR:

There is an error in the last decimal place calculated for $n<10$, $|\alpha|<10$, and $|x|<1$.

RUNNING TIME:

varies with all three arguments

ARGUMENT(S):

x in d
α in c
n in D

FUNCTION(S) OR RESULT(S):

$L_n^{(\alpha)}(X)$ is left in b

X, α, and n are undisturbed.

PROGRAM LISTING:

```
C ↓
d -
F •
B ↑
b ↕
c ↑
a V
c ↓
C •
B X
B ↕
c ↓
F •
c ↕
c •
C •
d -
b X
B -
c +
b ↕
B ↕
D ↓
c -
A S
A V

1  F ◊
```

SAMPLE RUN:

		V
n	5	S
	0.5	S
x	0.5	S
$L_n^{(\alpha)}(x) =$	-0.24375000000002	b◊
	12	S
	0	S
	10	S
	-9.90374645930200	b◊
	6	S
	1	S
	1	S
	-1.51527777777775	b◊
	6	S
	5	S
	5	S
	7.03472222222225	b◊
	3	S
	2	S
	1	S
	2.33333333333333	b◊
	8	S
	0.1	S
	1.1	S
	0.25845512152779	b◊